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Abhandlungen

Concentration Risk under Pillar 2: When are Credit Portfolios Infinitely Fine Grained?

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I. Introduction

There have been significant advances in analytical approaches to credit risk modeling since the first proposal of the new capital adequacy framework (Basel II) has been published in 1999 and it has been finalized in 2004/2006 by the *Basel Committee On Banking Supervision* (*BCBS*).[1] In the supervisory capital rules for portfolio credit risk a closed form solution for the measures of risk like Value at Risk (VaR) and Expected Loss (EL) has been achieved in the Internal Ratings Based (IRB) Approach for quantifying credit risk in Pillar 1 of Basel II. Such a model avoids time consuming Monte-Carlo methods as described in *Marrison* (2002) and is widely used in credit portfolio models.[2] Furthermore, such analytical models like the IRB-approach[3] also add benefit to the bank's credit risk management because the risk contribution of each exposure to the portfolio risk can be identified easily and additional approaches for risk-capital allocation as proposed by *Overbeck/Stahl* (2003) are not needed. However, several components of credit risk are not covered sufficiently by this analytical model. One of the most important "non-disclosures" seems to be (credit) concentration risk that is primarily considered in Pillar 2 of Basel II, since the *BCBS* might be aware of such shortcomings when using an analytical framework.[4] Particularly, it is stated in Basel II

[1] See *Basel Committee On Banking Supervision* (1999, 2001, 2003, 2005a, 2006).

[2] Particularly, Monte-Carlo simulation is used in the commercial models of CreditPortfolioView™, see *Wilson* (1997a, b), and CreditMetrics™, see *Gupton/Finger/Bathia* (1997).

[3] For the general scientific background of the IRB-model as an analytic credit risk model please refer to *Gordy* (2003) or *Finger* (2001).

[4] One of the main tasks of Pillar 2 is to ascertain, that "*risks [...] not fully captured by the Pillar 1 process (e.g. credit concentration risk)*" should be captured

that dealing with concentration risk seems to be one of the most important future tasks. Since concentration risk is not covered in Pillar 1, its quantification and management are two of the important problems in credit risk management under Pillar 2.[5]

The *BCBS* mainly distinguishes two sorts of concentration risk that emerge from[6]

(I) "*significant exposures to an individual counterparty or group of related counterparties*" and

(II) "*credit exposures to counterparties in the same economic sector or geographic region*".

The concentration risk (I) evolves from "single-name" concentrations. The BCBS distinguishes between two sorts of "single-name" concentrations.[7] One type of concentration risk pertains an exposure to one firm or to a conglomerate of economic highly dependent firms[8] that is extremely large compared to the rest of the exposures of the portfolio. In such a situation the default risk of the portfolio is mainly driven by the probability of default of this individual debtor. We call this type of risk "individual single-name" concentration risk. *Tasche/Theiler* (2004) and *Emmer/Tasche* (2005) integrated this type of risk in a IRB type model and concluded that consideration of an "individual single-name" concentration risk leads to spurious results when using VaR as a risk measure. Against this background we focus on the second type of "single-name" concentration that occurs if the bank holds a risk bucket (or sub-portfolio) containing a relatively small number of firms, each of them with large exposures. Such a risk bucket is hardly diversified because of the quite small number of debtors. Thus, a bank faces high losses if a large number of defaults appears, even if they are accidentally and not driven by default correlation of the firms. This type of concentration risk (I) can

adequately by the banks risk management using advanced methods, see *Basel Committee On Banking Supervision* (2005), paragraph 724.

[5] For further issues on concentration risk see *Basel Committee On Banking Supervision* (2005), paragraphs 770–777, as well as *Basel Committee On Banking Supervision* (2000), principle 12 (paragraphs 65–68) including the Appendix to Concentrations, paragraphs 2–6.

[6] See *Basel Committee On Banking Supervision* (2005), paragraph 773. For a general explanation of concentration risk under Basel II one may refer to Deutsche Bundesbank (2006) as well (only German language).

[7] See *Basel Committee On Banking Supervision* (2005c, 2005d). Some of the literature referred there will be discussed later on.

[8] Under Basel II such a conglomerate is called "connected group", see *Basel Committee On Banking Supervision* (2005a), paragraph 423.

be denoted as "portfolio single-name" concentration risk. It is linked with concentration risk (II) that arises from "sector" concentration. "Sector" concentration risk occurs if defaultable claims belong to a single sector with high default correlation. Though such a concentrated credit portfolio might incorporate many debtors, these debtors might default at the same time due to sector concentration and resulting high default correlation. The major problem caused by this form of "sector" concentration risk is the possibility of the false conclusion that such a large risk bucket (or sub-portfolio) is diversified.

However, such concentration risks are not just a Pillar 2 problem. Indeed, the existence of "portfolio single-name" and "sector" concentration risk of Pillar 2 directly lead to misspecified assumptions of analytic VaR-models like the *Merton*-type model of *Vasicek* (1987, 1991, 2002), that builds the bottom of the Internal Ratings Based (IRB) model of Basel II for quantifying credit risk under Pillar 1 and has become one of the standards in analytical credit portfolio modeling.[9] In the *Vasicek* model, portfolio credit risk is mainly quantified due to its default rate using the VaR as the risk measure. To achieve analytical tractability of the model, a so-called Asymptotic Single Risk Factor (ASRF) framework as explained in *Gordy* (2003) or *Bank/Lawrenz* (2003) is assumed. That is,

(A) the portfolio is infinitely fine grained and thus it consists of a nearly infinite number of credits with small exposures, and

(B) only one systematic risk factor influences the default risk of all loans in the portfolio.

Because of these two simplifications the measured VaR is portfolio-invariant, i.e. only the risk contribution of each defaultable claim to the systematic risk factor is of interest. Each individual claim does not cause any (further) diversification effect, since the portfolio already reached the highest possible degree of diversification. Unfortunately, both assumptions (A) and (B) are mutually exclusive in practice. Precisely, due to the limited factorization (assumption (B)) the model is only designed for small risk buckets, like rating grades as in *Gordy* (2000) or industry sectors as in *Rösch* (2003), rather than for whole credit portfolios. Hence, if one aims to meet assumption (B), the risk bucket under consideration is likely to consist of only a small number of loans. Resulting from this limitation, assumption (A) will become critical.

[9] See *Merton* (1974). It is also known as the one factor two state approach of CreditMetrics™, see *Finger* (1999). For the adoption in Basel II see *Finger* (2001) and additionally *Basel Committee On Banking Supervision* (2005b).

However, the violation of (A) or (B) do not have to lead to the fact, that the ASRF-framework can not be used at all for credit risk quantification. But one has to consider the consequences of the violation, i.e. the existence of concentration risk. Concretely, if assumption (A) is not met, we should account for concentration risk (I) especially in form of "portfolio single-name" concentration risk. However, if we exaggerate a risk bucket in order to meet assumption (A), assumption (B) is possibly violated and we should keep the existence of ("sector") concentration risk (II) in mind. This issue is not only a problem that should be accounted for in credit risk management when dealing with analytical models, but it is also critical for supervisory capital measurement in banks.[10] This raises the following question: Do the assumptions (A) and (B) of (the IRB-model under) Pillar 1 generally hold for our portfolio or do we have to quantify concentration risk for Pillar 2? Resulting from this, we identify two important tasks regarding risk concentrations:

(i) In which cases are the assumptions of the ASRF framework of the *Vasicek* model critical concerning the credit portfolio size?

(ii) In which cases are currently discussed adjustments for the VaR-measurement able to overcome the shortcomings of the *Vasicek* model?

The answers of both questions are available if we know the minimum number of loans in a risk bucket that is necessary in order to fulfill the granularity assumption (A) to achieve a required accuracy, say 5 %, of the analytical determined VaR in comparison to the true VaR using the *Vasicek* framework.[11] Unfortunately, numerical analyses on that topic are scarce. Thus, firstly, we oppose the existing formulas for the VaR using the granularity adjustment assuming a coarse grained, an (infinitely) fine grained as well as a medium grained portfolio. Additionally, we extend the existent framework in order to account for small sized portfolios.[12]

[10] Another solution to the problem of the violation of assumption (A) or (B) might be to cancel risk quantification under the IRB-approach and use internal models. However, this solution is not designated in Basel II.

[11] This question is also interesting when analysing the Basel II formula, because the designated add-on factor for the potential violation of assumption (A) was cancelled from the second consultative document to the third consultative document, see *Basel Committee On Banking Supervision* (2001, 2003). Thus, we only prove, under which conditions the assumption (A) of the *Vasicek* model is fulfilled. Of course, this model may suffer from other assumptions like the distributional assumption of standardised returns. However, since we only would like to address the topic of concentration risk, our focus should be reasonable. Additionally, the distributional assumptions seem not to have a deep impact on the measured VaR, see *Koyluoglu/Hickman* (1998a, b), *Gordy* (2000) or *Hamerle/Rösch* (2004, 2005a, b).

Secondly, we numerically infer the minimum number of loans in a portfolio using two definitions of accuracy in order to enhance the theoretical background with concrete facts on critical portfolio sizes.[13] This could give an advice which sub-portfolios have significant risk concentrations and thus should be controlled on credit portfolio and not on individual credit level. Like in the *Vasicek* model, we focus on gross loss rates in homogeneous credit portfolios, i. e. each borrower has an identical probability of default as well as an identical credit exposure and the loss rate is equal to one.[14] Furthermore we examine the granularity adjustment of an inhomogeneous portfolio based on a simulation as well. With our analysis we explain more about differences between simulated and analytically determined solutions to credit portfolio risk as well as between Basel II capital requirements for Pillar 2 with respect to Pillar 1.[15]

The rest of the paper is outlined as follows. In section II we briefly discuss the ongoing research and solutions in the field of measuring especially “single name” concentration risk in analytic credit portfolio models. In section III we describe the *Vasicek* model and derive the adjustment for small and medium sized risk buckets. The numerical analyses on homogeneous as well as on non-homogeneous risk buckets will be

[12] We motivate this procedure by the fact, that for market risk quantification of nonlinear exposures two factors of the *Taylor* series (fist and second order) are common to achieve a higher accuracy, see e. g. *Crouhy/Galai/ Mark* (2001) or *Jorion* (2003). This might be appropriate for credit risk as well. Furthermore, the higher order derivatives of VaR given by *Wilde* (2003) make it possible to systematically derive such a formula, which was already mentioned by *Gordy* (2004), but neither derived nor tested so far.

[13] BCBS already stated that in principle the effect of portfolio size on credit risk is well understood, but lacks practical analysis, see *Basel Committee On Banking Supervision* (2005c).

[14] Precisely, we assume non-stochastic loss rates. This might be satisfied by the fact that the number of defaults in a portfolio is still of main interest and in the Foundation IRB approach the loss rate is fixed for banks anyway. However, we finally examine the granularity adjustment of an inhomogeneous portfolio as well. Our setup is comparable to the one of *Cespedes/Herrero/Kreinin/Rosen* (2005), who analyse the sector concentration effect assuming infinitely fine grained risk buckets.

[15] Additionally, our article makes contribution to the ongoing research on analysing differences between Basel II capital requirements and banks internal “true” risk capital measurement approaches. Since the approximation of the regulatory capital requirements and the perceived risk capital of banks internal estimates for portfolio credit risk is often stated as the major benefit of Basel II, see e. g. *Hahn* (2005), p. 127, but often not observed in practice (see e.g. exemplary calculations for real world portfolios of SunGard Data Systems Inc.), this task might be of relevance in future.

taken out in section IV. Section V summarizes the results and points out some key issues on the use of the IRB-model of Basel II for credit risk management.

II. Literature on Portfolio Concentrations in the *Vasicek* Model

The principle of incorporating the effect of the portfolio size in the analytical *Vasicek* model discussed in literature is very simple. As a first step it is assumed, that the portfolio is infinitely fine grained and the VaR can be determined under the ASRF framework. However, an add-on factor is constructed, that accounts for the finite size of the portfolio and that converges to zero if the assumption (A) of infinite granularity is (nearly) met. A version of this so-called granularity adjustment was part of Basel II until the second consultative document,[16] but because of some theoretical shortcomings of this model, a more convenient formula for the adjustment was presented by *Wilde* (2001).[17] Precisely, this factor equals the first element different from zero that results from a *Taylor* series expansion of the VaR around the ASRF solution.[18] However, a concrete number of loans that is required to meet a pre-defined accuracy interval for the VaR (including the granularity adjustment) is not discussed widely. *Gordy* (2003) comes to the conclusion that the granularity adjustment works fine for risk buckets of more than 200 loans considering low credit quality buckets and for more than 1000 loans for high credit quality buckets. However, he uses the CreditRisk$^+$ framework from *Credit Suisse Financial Products* (1997) and not the *Vasicek* model that builds the basis of Basel II, and he does not analyse the effect of different correlation factors as they are assumed in Basel II. Additionally, solutions for further improvement of the granularity are postulated in the literature without examining the results.[19]

[16] The effectiveness and the eligibility of the (cancellation of the) granularity add-on from the second to the third consultative document of Basel II is only discussed vaguely in the literature so far, see e.g. *Bank/Lawrenz* (2003), p. 543.

[17] The main criticism of the formula in Basel II was that the granularity adjustment was derived via the CreditRisk$^+$ methodology, whereas *Wilde* (2001) was able to derive a formula consistent with the *Vasicek* model.

[18] For the derivation of the granularity adjustment in the *Vasicek* model see also *Pykhtin/Dev* (2002) as well as *Pykhtin* (2004). The derivation of the granularity adjustment by a *Taylor* series expansion is mainly motivated by *Gordy* (2004) and *Rau-Bredow* (2002/2004) and we come to that in section IV. Additionally, *Martin/Wilde* (2002) show that via the heat equation the same results can be achieved whereas the saddle point method agrees only in special cases, e.g. CreditRisk$^+$ with one sector.

Finally, *Pykhtin* (2004) recently extended the analytical VaR derivation using a multi-factor adjustment in order to relax assumption (B). Due to a multi-factor layout of the model the observed risk buckets can be enlarged, so that the granularity-assumption (A) becomes less critical. Nonetheless, an additional adjustment would be needed and the analytical solution (and the parameter estimation) will become more complicated.[20] *Cespedes/Herrero/Kreinin/Rosen* (2005) presented a smart solution to reduce some of those troubles. They suggested to divide the portfolio under consideration into small risk buckets, so that each risk bucket belongs to one and only one sector.[21] Consequently, even if the ASRF framework is used, "sector" concentration risk (II) is not a problem. However, each risk bucket has to fulfill assumption (A) of infinite granularity. Therefore, from the practitioners' perspective it is interesting to know, which minimum size of the portfolio is needed to meet assumption (A) and how a granularity adjustment is able to improve the results.

III. Adjusting for Granularity in the *Vasicek* Model

1. Coarse and Fine Grained Risk Buckets

With reference to *Vasicek* (1987, 1991, 2002) and *Finger* (1999, 2001) we use a one-period one-factor model for determining the portfolio default rate of a homogeneous portfolio and its VaR.[22] Precisely, we observe a risk bucket of J obligors at $t = 0$ with respect to $t = T$. Each obligor $j \in \{1, \ldots, J\}$ holds an exposure of the amount $E_j = E$. The discrete time process of "normalized" returns[23] $\tilde{a}_{j,T}$ at $t = T$ of the assets of each obligor j is represented by the following one-factor model[24]

$$\tilde{a}_{j,T} = \sqrt{\varrho} \cdot \tilde{x}_T + \sqrt{1-\varrho} \cdot \tilde{\varepsilon}_{j,T} \, . \tag{1}$$

19 See footnote 12.

20 Especially the data requirements for estimating asset correlations in this multi-factor model are a big challenge.

21 At a stretch, the multiple (small) risk buckets are represented by a single systematic factor each. Since these systematic factors are not perfectly correlated, they present a solution for "sector" diversification, which is the opposite of the "sector" concentration. Thus, their view on the impact of sectors within a portfolio is slightly different from the one of the committee.

22 The following model outline is very similar to *Rösch* (2003).

23 The returns are normalized by subtracting the expected return and dividing the resulting term by the standard deviation in order to get standard normally distributed variables.

24 To keep track of the model, stochastic variables are marked with a tilde "~".

in which $\tilde{x}_T \sim N(0,1)$ and $\tilde{\varepsilon}_{j,T} \sim N(0,1)$ are i.i.d. with $j \in \{1,\ldots,J\}$, i.e. they are independent (and identical) normally distributed with mean zero and standard deviation one. Therefore, $\tilde{x}_T$ serves as the common shared, systematic factor that represents the overall economic condition of all obligors. Besides this, the risk factors $\tilde{\varepsilon}_{j,T}$ are the idiosyncratic factors, that are independent from the systematic factor and account for the individual risk of each borrower. The asset correlation ϱ between all borrowers is assumed to be constant in the risk bucket and also expresses the fraction of risk to the common shared factor measured by the variance. Additionally, we assume that the obligor j defaults at $t = T$ when its "normalized" return falls short of an exogenously given default threshold

$$b_{j,T} = N^{-1}(PD_j), \tag{2}$$

in which $N^{-1}(\cdot)$ stands for the inverse cumulative standard normal distribution and PD_j defines the (unconditional) probability of default of obligor j. Due to homogeneity we set $PD_j = PD$ and thus $b_{j,T} = b_T$ for all $j \in \{1,\ldots,J\}$. Conditional on a realisation of the systematic factor the probability of default of each obligor is[25]

$$P\big(\tilde{a}_{j,T} < b_T | \tilde{x}_T\big) = E\big(I\big(\tilde{a}_{j,T} < b_T | \tilde{x}_T\big)\big) = N\left(\frac{N^{-1}(PD) - \sqrt{\varrho}\cdot\tilde{x}_T}{\sqrt{1-\varrho}}\right) =: p(\tilde{x}_T) \tag{3}$$

in which $I(\cdot)$ represents the indicator function that is 1 in the event of default and 0 in case of survival of the obligor and $N(\cdot)$ stands for the cumulative standard normal distribution. Since conditional on a realization $\tilde{x}_T = x_T$ the individual default events are independent, the (conditional, still uncertain) number of defaults $\tilde{K}_T|x_T$ (and the gross loss rate) of the portfolio are binomial distributed with the probability $p(x_T)$, i.e.

$$\tilde{K}_T | x_T \sim B\big(J;\, p(x_T)\big). \tag{4}$$

With reference to *Vasicek* (1987), see also *Gordy/Heitfield* (2000), we are able to calculate the unconditional probability of having k_T defaults and we get

$$P\left(\tilde{D}_T = \frac{k_T}{J}\right) = \int_{-\infty}^{+\infty} \binom{J}{k_T} \cdot p(x_T)^{k_T} \cdot \big(1 - p(x_T)\big)^{J-k_T} \cdot dN(x_T) \tag{5}$$

where $\tilde{D}_T$ marks the (uncertain) portfolio gross loss rate.

[25] In the following "P" denotes the probability of an event and "E" stands for the expectation operator.

For risk quantification we use the VaR on confidence level z of the observed risk bucket, that is the z-quantile q_z of the loss variable, in which $z \in (0{,}1)$ is the target solvency probability. Precisely, like *Gordy* (2004), we define the VaR as the loss that is only exceeded with the probability of at most $1 - z$, i. e.

$$\text{(6)} \qquad \mathrm{VaR}_z(\tilde{D}_T) := q_z(\tilde{D}_T) := \inf\left(d_T : P(\tilde{D}_T \le d_T) \ge z\right).$$

With respect to equation (5) we get

$$\text{(7)} \qquad \mathrm{VaR}_z^{(cg)}(\tilde{D}_T) = \inf\left(d_T : P(\tilde{D}_T \le d_T) = \sum_{k_T = 1}^{d_T \cdot J} P\left(\frac{k_T}{J}\right) \ge z\right)$$

for the VaR of the risk bucket. We call this the VaR of a coarse grained (homogeneous) bucket, since this formula is valid for any bucket size J. Thus, the granularity assumption (A) of section I is not considered in this situation. The result of expression (7) can only be derived numerically.

As a next step we apply the concept of an (infinitely) fine grained portfolio, i. e. we assume an infinite number of obligors in the risk bucket and the weight of each exposure shrinks to zero,[26] i. e.

$$\text{(8)} \qquad \lim_{J \to \infty} \sum_{j=1}^{J} w_j^2 = 0 \quad \text{with} \quad w_j = E_j \Big/ \sum_{k=1}^{J} E_k \overset{E_j = E_k = E}{=} \frac{1}{J}.$$

For the VaR of the portfolio gross loss rate according to *Vasicek* (2002) or *Bluhm/Overbeck/Wagner* (2003) we receive

$$\text{(9)} \qquad \begin{aligned} \lim_{J \to \infty} \mathrm{VaR}_z^{(cg)}(\tilde{D}_T) &=: \mathrm{VaR}_z^{(fg)}(\tilde{D}_T) = \mathrm{VaR}_z^{(fg)}\left(E(\tilde{D}_T | \tilde{x}_T)\right) \\ &= N\left(\frac{N^{-1}(PD) - \sqrt{\varrho} \cdot q_{1-z}(\tilde{x}_T)}{\sqrt{1-\varrho}}\right), \end{aligned}$$

in which $q_{1-z}(\tilde{x}_T)$ stands for the $(1 - z)$-quantile of the systematic factor. This is the (well established) VaR-figure of an (infinitely) fine grained risk bucket and it is equal to the expected loss rate as defined in equation (3) conditional on $q_{1-z}(\tilde{x}_T)$. Obviously, the credit risk only relies on the systematic factor, since due to the infinite number of exposures the idiosyncratic risks associated with each individual obligor cancel each

[26] Here we used the assumption due to *Vasicek* (2002), p. 160, that can be derived from the assumption due to *Bluhm/Overbeck/Wagner* (2003), p. 87, by using *Kronecker*s Lemma.

other out and are diversified completely. However, in a real-world application assumption (8) surely not holds and a fraction of risk, that comes from the idiosyncratic factors, stays in the bucket.

2. *Small and Medium Sized Risk Buckets*

In this section we present two adjustments for the VaR formula (9) to take into account that in real world portfolios the idiosyncratic risk can not be diversified completely. The first formula was derived by *Wilde* (2001), the second is an extension and will be developed below. These adjustments can be derived as a *Taylor* series expansion of VaR around the ASRF solution.[27] Precisely, we subdivide the portfolio loss rate into a systematic and an idiosyncratic part, i. e.

$$\tilde{D}_T = E(\tilde{D}_T|\tilde{x}_T) + \left[\tilde{D}_T - E(\tilde{D}_T|\tilde{x}_T)\right] =: \tilde{Y} + \lambda\tilde{Z}. \tag{10}$$

Thus, the first term $E(\tilde{D}_T|\tilde{x}_T) =: \tilde{Y}$ describes the systematic part of the portfolio loss rate that can be expressed as the expected loss rate conditional on $\tilde{x}_T$ (see also equation (3) and (9)). The second term $\tilde{D}_T - E(\tilde{D}_T|\tilde{x}_T) =: \lambda\tilde{Z}$ of equation (10) stands for the idiosyncratic part of the portfolio loss rate. Therefore, $\tilde{Z}$ describes the general idiosyncratic component and λ decides on the fraction of the idiosyncratic risk that stays in the portfolio. Obviously, λ tends to zero if the number of obligors J converges to infinity, since this fraction (of the idiosyncratic risk) vanishes if the granularity assumption (A) from section I holds. However, for a granularity adjustment we claim that the portfolio is only "nearly" infinitely granular and thus λ is just close to but exceeds zero. In order to incorporate the idiosyncratic part of the portfolio loss rate into the VaR-formula we perform a *Taylor* series expansion around the systematic loss at $\lambda = 0$. We get

$$\begin{aligned}\mathrm{VaR}_z(\tilde{D}_T) &= \mathrm{VaR}_z\big(\tilde{Y}+\lambda\tilde{Z}\big) = \mathrm{VaR}_z(\tilde{Y}) + \lambda\left[\frac{\partial \mathrm{VaR}_z\big(\tilde{Y}+\lambda\tilde{Z}\big)}{\partial\lambda}\right]_{\lambda=0} \\ &\quad + \frac{\lambda^2}{2!}\left[\frac{\partial^2 \mathrm{VaR}_z\big(\tilde{Y}+\lambda\tilde{Z}\big)}{\partial\lambda^2}\right]_{\lambda=0} + \frac{\lambda^3}{3!}\left[\frac{\partial^3 \mathrm{VaR}_z\big(\tilde{Y}+\lambda\tilde{Z}\big)}{\partial\lambda^3}\right]_{\lambda=0} + \ \cdots \ .\end{aligned} \tag{11}$$

Thus, the first term describes the systematic part of the VaR and all other terms add an additional fraction to the VaR due to the undiversified idio-

[27] The concept of this approach can be compared with the derivation of the Duration/Convexity in the context of bond management.

syncratic component. For the granularity adjustment it turns out, that only the terms of the order two and higher are non-zero.

To compute the elements of the *Taylor* series, we require the derivatives of VaR. With reference to *Wilde* (2003), the formula for the first five derivatives (m = 1, 2, …, 5) of VaR in this context is given as[28]

$$\frac{\partial^m \mathrm{VaR}_z(\tilde{Y}+\lambda\tilde{Z})}{\partial\lambda^m} = (-1)^m \frac{1}{f_Y}\left[-\frac{d^{m-1}}{dl^{m-1}}\left(\mu_m \cdot f_Y\right) + \alpha(m)\frac{d}{dx}\left(\frac{1}{f_Y}\frac{d}{dl}\left(\mu_2 \cdot f_Y\right) \cdot \frac{d^{m-3}}{dl^{m-3}}\left(\mu_{m-2} \cdot f_Y\right)\right)\right]\Bigg|_{l=\mathrm{VaR}_z(\tilde{Y})}, \tag{12}$$

with $\alpha(1) = \alpha(2) = 0$, $\alpha(3) = 1$, $\alpha(4) = 3$ and $\alpha(5) = 10$. Here f_Y is the density function of the systematic loss rate of the risk bucket and μ_m stands for the m^{th} (conditional) moment about the origin of the loss rate conditional on the systematic factor.

Concurrently, the first derivative of VaR equals zero,[29] so that the second derivative is the first relevant element underlying the granularity adjustment. With reference to *Wilde* (2001) and *Rau-Bredow* (2002) the *Taylor* series expansion up to this quadratic term leads to the following formula for the VaR including the granularity adjustment, that is[30]

$$\begin{aligned} &\mathrm{VaR}_z^{(1.\,\mathrm{Order\,Adj.})} = \mathrm{VaR}_z^{(fg)} + \triangle l_1 \quad \text{with} \\ &\triangle l_1 = -\frac{1}{2\,n(x)}\frac{\partial}{\partial x}\left(\frac{n(x)\cdot\, V\left[\tilde{D}_T|\tilde{x}=x\right]}{\frac{d}{dx}E\left[\tilde{D}_T|\tilde{x}=x\right]}\right)\Bigg|_{x=q_{1-z}(\tilde{x}_T)}, \end{aligned} \tag{13}$$

in which $n(x)$ describes the standard normal density function at x. Thus, the VaR figure of the infinitely fine grained portfolio due to equation (9) is adjusted by an additional term, that is the first term different from zero of the *Taylor* series expansion (11). We call this expression the ASRF solution with first order (granularity) adjustment. Under the condition of the *Vasicek* model, particularly the probability of default is assumed to be given by formula (3), we receive for the granularity add-on of a homogeneous portfolio[31]

[28] The first two derivatives were already presented by *Gourieroux/Laurent/Scaillet* (2000). *Wilde* (2003) presents a general formula for all derivatives of VaR. For our derivation the stated formula is sufficient.

[29] This is valid because the added risk of the portfolio is unsystematic; see *Martin/Wilde* (2002) for further explanations.

[30] "V" denotes the variance operator.

[31] In Appendix A an analogous formula is stated for inhomogeneous portfolios.

$$\triangle l_1 = \frac{1}{2J}\left(\left(1-N(y)\right)\left[\frac{N(y)}{n(y)}\frac{q_{1-z}(\tilde{x}_T)\cdot\left(1-2\varrho\right)-N^{-1}(PD)\sqrt{\varrho}}{\sqrt{\varrho}\sqrt{1-\varrho}}\ -\ 1\right]\right.$$
$$\left. + N(y)\right)\Bigg|_{y=\frac{N^{-1}(PD)-\sqrt{\varrho}\,q_{1-z}\left(\tilde{x}_T\right)}{\sqrt{1-\varrho}}}, \tag{14}$$

that is the formula presented by *Pykhtin/Dev* (2002) in the special case that we only model the gross loss rates. Obviously, the additional term is of order $O(1/J)$[32], that is in itself an asymptotic result, meaning that higher order terms are neglected.

Summing up both analytically derived formulas (9) and (13) for the VaR, the ASRF solution might only be exact if the term (14) of order $O(1/J)$ is close to zero, whereas the ASRF solution including the first order granularity adjustment might only be sufficient if the terms of order $O(1/J^2)$ vanish. For medium sized risk buckets this might be true, but if the number of credits in the portfolio is getting considerably small, an additional factor might be appropriate. Particularly, the mentioned granularity adjustment is linear in $1/J$ and this might not hold for small portfolios. Indeed, *Gordy* (2003) shows by simulation, that the portfolio loss seems to follow a concave function and therefore the adjustment (14) would slightly overshoot the theoretically optimal add-on for smaller portfolios.[33]

An explanation of the described behaviour is that the first order adjustment only takes the conditional variance into account, whereas higher conditional moments are ignored, which result from the higher order terms (see the derivatives in equation (12)). With the intention to improve the adjustment for small portfolio sizes, now the $O(1/J^2)$term will be derived and thus the error will be reduced to $O(1/J^3)$.[34] Having a closer look at the derivatives of VaR, the fourth and a part of the fifth element of the *Taylor* series can be identified to be relevant for the $O(1/J^2)$ terms.[35] Using the methodology of formula (11) this yields to the following term

$$\mathrm{VaR}_z^{(1.+2.\,\mathrm{Order\,Adj.})} = \mathrm{VaR}_z^{(fg)} + \triangle l_1 + \triangle l_2 \tag{15}$$

with

[32] The *Landau* symbol $O(\cdot)$ is defined as in *Billingsley* (1995), A18.
[33] *Gordy* (2003) observes the concavity of the granularity add-on for a high-quality portfolio (A-rated) up to a portfolio size of 1,000 debtors.
[34] See *Gordy* (2004), p. 112, footnote 5, for a similar suggestion.
[35] See Appendix B for details.

$$\triangle l_2 = \frac{1}{6\,n(x)}\frac{d}{dx}\left(\frac{1}{d\mu_1(x)/dx}\frac{d}{dx}\left[\frac{\eta_3(x)\cdot n(x)}{d\mu_1(x)/dx}\right]\right) + \frac{1}{8\,n(x)}\frac{d}{dx}\left[\frac{1}{n(x)}\frac{1}{d\mu_1(x)/dx}\left(\frac{d}{dx}\left[\frac{\eta_2(x)\cdot n(x)}{d\mu_1(x)/dx}\right]\right)^2\right]\Bigg|_{x=\mathrm{VaR}_{1-z}(\tilde{x})} \tag{16}$$

in which $\mu_1(x) = E(\tilde{D}_T|\tilde{x} = x)$ is the 1^{st} (conditional) moment about the origin and $\eta_m(x) = \eta_m(\tilde{D}_T|\tilde{x} = x)$ is the m^{th} (conditional) moment about the mean. In the context of the *Vasicek* model and under consideration of homogeneity we receive for this second add-on factor[36]

$$\begin{aligned}\triangle l_2 = {} & \frac{1}{6J^2s^2n_y^2}\left[\left(x^2-1+s^2+3x\,s\,y+2s^2y^2\right)\left(N_y-3N_y^2+2N_y^3\right)\right. \\ & + \; s\,n_y(2x+3s\,y)\left(1-6N_y+6N_y^2\right) - s^2n_y\left(y-6\left[N_yy-n_y\right]+6N_y\left[N_yy-2n_y\right]\right)\Big] \\ & - \; \frac{1}{8J^2s^3n_y^3}\left[(-x-3s\,y)\left(\left[N_y-N_y^2\right]\left[-x-s\,y\right]-s\,n_y\left[1-2N_y\right]\right)^2\right. \\ & + \; 2\left(\left[N_y-N_y^2\right]\left[x+s\,y\right]+s\,n_y\left[1-2N_y\right]\right) \\ & \cdot \left(\left[N_y-N_y^2\right]\left[1-s^2\right]-s\,n_y\left[1-2N_y\right]\left[x+s\,y\right]+s^2n_y\left[y+2\left(n_y-N_yy\right)\right]\right)\Big],\end{aligned} \tag{17}$$

with $N_y = N(y)$, $n_y = n(y)$, $y = \dfrac{N^{-1}(PD)-\sqrt{\varrho}\cdot x}{\sqrt{1-\varrho}}$, $s = \dfrac{\sqrt{\varrho}}{\sqrt{1-\varrho}}$, and $x = q_{1-z}(\tilde{x}_T)$.

Thus, the additional term is of order $O(1/J^2)$ and equation (15) for the VaR only neglects terms of order $O(1/J^3)$. We will refer to this expression as the VaR under the ASRF solution with (first and) second order granularity adjustment. In terms of numbers of credits the error is reduced in the postulated way. Even if the formulas appear quite complex, both adjustments are easy to implement, fast to compute and we do not have to run Monte Carlo simulations and thereby avoid simulation noise.

IV. Numerical Analysis of Granularity

1. *The Impact of the Approximations on the Portfolio Quantile*

For a detailed analysis of the granularity assumption (A) as mentioned in section I, we firstly would like to discuss the general behaviour of the four procedures for risk quantification of homogeneous portfolios presented in section III.1 and section III.2, that are

[36] See Appendix C for the derivation of the more general inhomogeneous case.

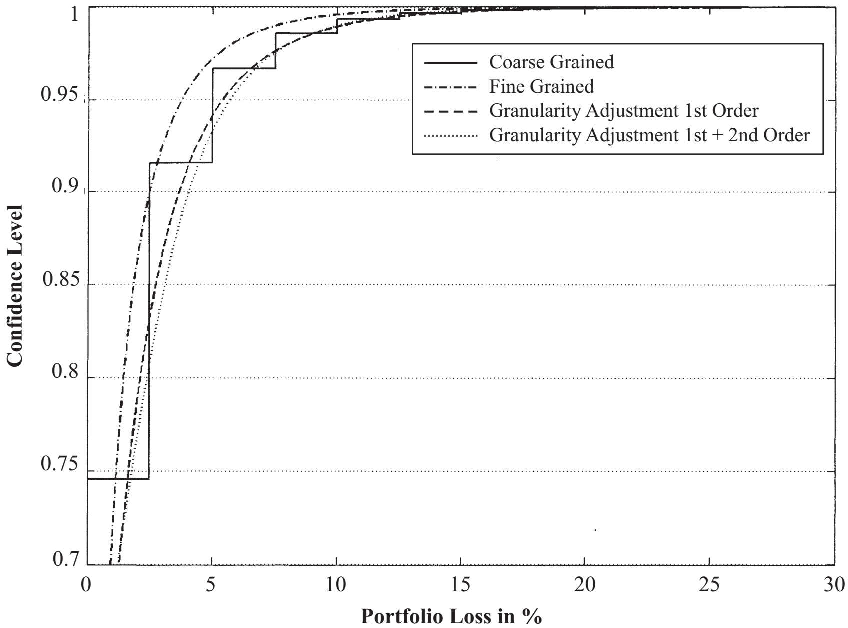

Figure 1: Distribution of Losses for a Wide Range of Probabilities

(a) the numerically "exact" coarse grained solution (see equation (5))

(b) the fine grained ASRF solution (see equation (9))

(c) the ASRF solution with first order adjustment (see equations (13) and (14))

(d) the ASRF solution with first and second order adjustments (see equations (14) to (17))

Therefore, we evaluate the portfolio loss distribution of a simple portfolio, that consists of 40 credits, each with a probability of default of $PD = 1\%$. We set the correlation parameter to $\varrho = 20\%$.[37] Using these parameters, we calculate the loss distribution using the "exact" solution (a) as well as the approximations (b) to (d). The results are shown in Figure 1 for portfolio losses up to 30 % (12 credits) and the corresponding

[37] The chosen portfolio exhibits high unsystematic risk and therefore serves as a good example in order to explain the differences of the four solutions. However, we evaluated several portfolios and the results do not differ widely. Additionally, we claim that the general statements can also be applied to heterogeneous portfolios as well.

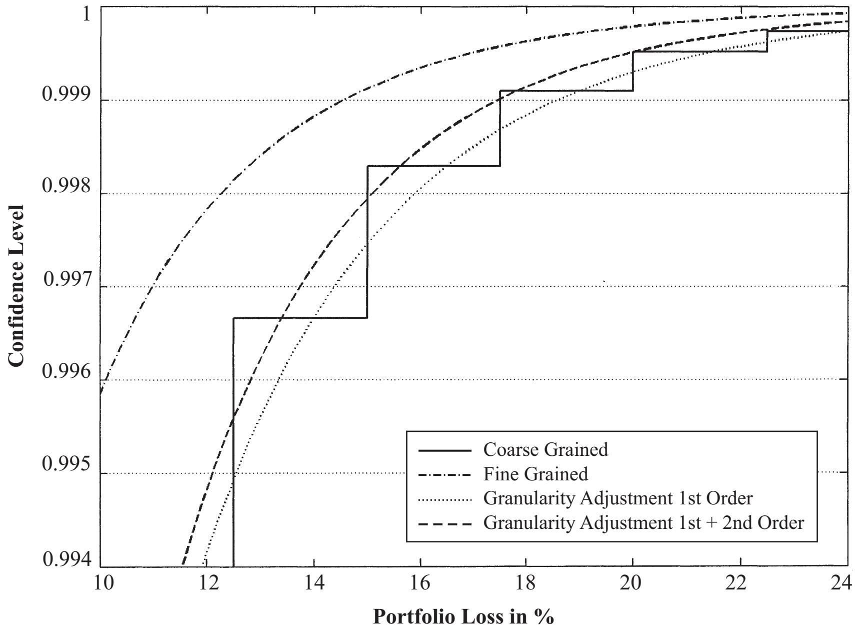

Figure 2: Distribution of Losses for High Confidence Levels

quantiles (of the loss distribution) starting at 0.7. See Figure 2 for the region of high quantiles from 0.994 on, that are of special interest in a VaR-framework for credit risk with high confidence levels.

It is obvious to see that the coarse grained solution (a) is not continuous, since the distribution of defaults is a discrete binomial mixture, whereas all other solutions (b) to (d) are "smooth" functions. This is caused by the fact, that these approximations for the loss distribution assume an infinitely granular portfolio, i.e. the loss distribution is monotonous increasing and differentiable (solution (b)), or at least are derived from such an idealized portfolio ((c) and (d)).

Firstly, we may examine the result for the VaR-figures at confidence levels 0.995 and 0.999. Using the exact, discrete solution (a) the VaR is 12.5% (or 5 credits) for the 0.995 quantile and 17.5% (or 7 credits) for the 0.999 quantile. Compared to this, the ASRF solution (b) exhibits significant lower loss rates at these confidence levels, that are 9.46% for the 0.995 quantile and 14.55% for the 0.999 quantile. Obviously, the ASRF solution underestimates the loss rate, since it does not take (additional) concentration risks into account.

If we add the first order adjustment (c), the VaR figures increase compared to the ASRF solution (b) with values 12.55 % for the 0.995 quantile and 18.59 % for the 0.999 quantile. Both values are good proxies for the "true" solution (a). Especially the VaR at 0.995 confidence level is nearly exact (12.55 % compared to 12.5 %). However, (c) seems to be a conservative measure, since the VaR is positively biased. Using the additional second order adjustment (d), the VaR lowers to 12.12 % for the 0.995 quantile and 17.48 % for the 0.999 quantile. In this case the VaR at 0.999 confidence level is nearly exact (17.48 % compared to 17.5 %). Nonetheless, (d) is likely to be a progressive approximation for the "true" solution (a), since the VaR is negatively biased.

Summing up the results from our experience (see also Figure 1 and Figure 2), using the ASRF solution (b) the portfolio distributions shift to lower loss rates for the VaR compared to the "exact" solution (a), since an infinitely high number of credits is presumed. Precisely, the idiosyncratic risk is diversified completely, resulting in a lower portfolio loss rate at high confidence levels. If one incorporates the first order granularity adjustment (c), this effect will be weakened and especially for the relevant high confidence levels the portfolio loss rate will increase compared to the ASRF solution (b). This means, that the first order granularity adjustment is usually positive.[38]

However, if the second order granularity adjustment (d) is added, the portfolio loss distribution will shift backwards again (for high confidence levels). This can be addressed to the alternating sign of the *Taylor* series as can be seen in formula (12). Since the first order granularity adjustment is positive, the second order adjustment tends to be negative. Summing up, with the incorporation of the second order adjustment (d) the approximation of the discrete distribution of the coarse grained portfolio (a) is (in general) less conservative compared to the (only) use of the first order adjustment. However, a clear conclusion, that the application of second order adjustment (d) in order to approximate the discrete numerical derived distribution (a) for high confidence levels outperforms the only use of the first order adjustment (c), can not be stated.[39]

To conclude, if we appraise the approximations for the coarse grained portfolio, we find both adjustments (c) and (d) to be a much better fit of the

[38] See *Rau-Bredow* (2005) for a counter-example for very unusual parameter values. This problem can be addressed to the use of VaR as a measure of risk which does not guarantee sub-additivity; see *Artzner/Delbaen/Eber/Heath* (1999).

[39] By contrast, we expected a significant enhancement by using the second order adjustment like mentioned in *Gordy* (2004), p. 112, footnote 5.

numerical solution in the (VaR relevant) tail region of the loss distribution than the ASRF solution, whereas the first order adjustment is more conservative and seems to give the better overall approximation in general.

2. *Size of Fine Grained Risk Buckets*

Reconsidering the assumptions of the ASRF framework (see section I), we found assumption (A) – the infinite granularity assumption – to be critical in a one factor model. Thus, we investigate in detail the critical numbers of credits in homogeneous portfolios that fulfill this condition.

Therefore, we firstly have to define a critical value for the derivation of the "true" VaR figure from solution (a) from the "idealized" VaR of the ASRF solution (b) to discriminate an infinite granular portfolio from a finite granular portfolio. We do that in two ways.

Firstly, one may argue, that the fine grained approximation (9) in order to calculate the VaR is only adequate, if its value does not exceed the "true" VaR from equation (7) of the coarse grained bucket minus a target tolerance β_T both using a confidence level of 0.999. Precisely, we define a critical number $\mathrm{I}_{c,per}^{(fg)}$ of credits in the bucket, so that each portfolio with a higher number of credits than $\mathrm{I}_{c,per}^{(fg)}$ will meet this specification. We use the expression[40]

$$\mathrm{I}_{c,per}^{(fg)} = \inf\left(J : \left|\frac{\mathrm{VaR}_{0.999}^{(fg)}(\tilde{D}_T)}{\mathrm{VaR}_{0.999}^{(cg)}(\tilde{D}_T = \tilde{K}_T/n)} - 1\right| < \beta_T \text{ for all } n \in \mathbf{N}^{\geq J}\right) \quad (18)$$
$$\text{with } \beta_T = 0.05.$$

Here, we set the target tolerance β_T to 5 %, meaning, that the "true" VaR specified by coarse grained risk buckets does not differ from the analytic VaR using the fine grained solution (9) by more than 5 %, if the number of credits in the bucket reaches at least $\mathrm{I}_{c,per}^{(fg)}$.

Secondly, the fine grained approximation (b) of the VaR ("idealized" VaR) may be sufficient as long as its result using a confidence level of 0.999 does not exceed the "true" VaR as defined by solution (a) of the coarse grained bucket using a confidence level of 0.995, i.e.

$$\mathrm{I}_{c,abs}^{(fg)} = \sup\left(J : \mathrm{VaR}_{0.999}^{(fg)}(\tilde{D}_T) < \mathrm{VaR}_{0.995}^{(cg)}\left(\tilde{D}_T = \frac{\tilde{K}_T}{J}\right)\right). \quad (19)$$

[40] To address to the minimum number after which the target tolerance will permanently hold, we have to add the notation "for all $n > J$" because the function of the coarse grained VaR exhibit jumps dependent on the number of credits.

Table 1

Critical Number of Credits from that ASRF Solution can be Stated to be Sufficient for Measuring the True VaR (see Formula (18))

	AAA up to AA- 0,03 %	A- up to A+ 0,05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
3.0 %	35986	23985	5389	5184	4105	3176	2057	1390	988	478	370	205
3,5 %	30501	20122	4627	4457	3544	2755	1801	1214	861	421	322	175
4.0 %	26051	17272	4054	3851	3076	2402	1563	1077	760	375	295	161
4,5 %	22372	14906	3569	3392	2719	2132	1398	958	690	350	271	145
5.0 %	19669	13160	3153	3047	2412	1928	1273	866	628	320	255	128
5,5 %	17723	11667	2840	2701	2180	1722	1145	784	564	289	229	125
6.0 %	15715	10590	2611	2442	1977	1566	1032	711	515	264	205	116
6,5 %	14276	9452	2366	2252	1828	1428	946	655	477	251	201	106
7.0 %	12730	8637	2148	2045	1665	1327	869	615	457	226	185	101
7,5 %	11633	7915	1990	1896	1547	1214	827	578	412	209	167	90
8.0 %	10657	7272	1813	1761	1414	1133	762	527	389	206	160	87
8,5 %	9785	6695	1720	1607	1318	1040	703	505	357	200	156	87
9.0 %	9222	6176	1571	1498	1231	992	660	460	338	183	143	80
9,5 %	8504	5707	1466	1427	1152	930	610	443	326	164	135	76
10.0 %	7853	5281	1399	1334	1079	873	597	419	304	157	132	68
10,5 %	7262	5015	1309	1249	1011	804	552	382	289	153	118	70
11.0 %	6900	4655	1226	1170	949	756	532	376	285	144	120	65
11,5 %	6398	4324	1149	1097	911	726	493	357	257	138	109	64
12.0 %	6099	4127	1103	1053	838	684	466	332	254	135	107	58
12,5 %	5669	3843	1036	989	806	645	450	315	242	127	103	60
13.0 %	5419	3677	974	952	759	622	435	299	226	117	94	53
13,5 %	5046	3430	915	896	732	587	395	284	211	117	98	55
14.0 %	4701	3290	882	843	706	555	391	288	201	110	87	52
14,5 %	4510	3073	851	794	666	536	362	263	200	101	91	50
15.0 %	4331	2954	822	767	629	519	344	250	195	108	84	51

	AAA up to AA- 0,03 %	A- up to A+ 0,05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
15,5 %	4044	2763	775	741	594	491	349	254	178	95	81	52
16.0 %	3892	2661	731	717	589	476	324	226	186	100	78	44
16,5 %	3748	2564	690	677	557	451	315	220	174	96	75	51
17.0 %	3507	2403	668	639	540	427	299	225	159	86	67	42
17,5 %	3383	2320	647	619	511	404	291	205	159	95	66	38
18.0 %	3167	2241	611	585	496	403	277	200	152	80	70	33
18,5 %	3060	2103	593	583	469	382	263	195	145	90	61	34
19.0 %	2959	2034	576	551	456	362	250	186	142	85	65	35
19,5 %	2863	1969	544	521	432	352	250	186	129	80	61	30
20.0 %	2685	1850	529	507	420	343	244	173	133	77	57	31
20,5 %	2601	1793	500	493	409	317	232	165	127	74	58	32
21.0 %	2522	1739	487	466	377	326	227	170	131	73	51	26
21,5 %	2446	1635	474	454	367	301	216	158	119	63	52	27
22.0 %	2297	1587	448	442	368	302	211	163	123	64	53	28
22,5 %	2230	1541	437	418	349	279	206	152	118	63	55	29
23.0 %	2167	1498	413	408	350	280	191	145	113	57	53	30
23,5 %	2036	1457	415	398	332	266	192	142	111	58	51	22
24.0 %	1980	1371	393	388	324	252	193	132	98	54	49	23

Corporates, Sovereigns, and Banks | SMEs (5 Mio. < Sales < 50 Mio.) | SMEs (Sales < 5 Mio.) | Mortgage | Revolving Retail | Other Retail

Table 2

Critical Number of Credits from that the Exact Solution on Confidence Level 0.995 Exceeds the Infinite Fine Granularity on Confidence Level 0.999 (see Formula (19))

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
3.0 %	5499	3885	997	1019	786	678	464	329	255	165	143	123
3,5 %	4354	3126	836	793	665	542	380	274	217	138	122	110
4.0 %	3428	2508	701	666	564	428	308	227	184	118	103	94
4,5 %	3111	1998	588	558	434	364	266	200	155	100	93	79
5.0 %	2436	1830	490	466	404	308	230	175	138	92	83	70
5,5 %	2239	1445	406	386	339	288	198	154	123	77	71	65
6.0 %	1724	1338	380	361	283	244	170	135	109	74	69	57
6,5 %	1599	1037	312	297	266	204	161	117	97	68	58	56
7.0 %	1489	968	294	280	220	193	138	112	85	62	57	50
7,5 %	1114	906	238	264	208	183	131	97	82	57	50	46
8.0 %	1044	681	225	214	197	152	111	93	72	52	46	42
8,5 %	982	641	214	204	161	145	106	80	63	47	45	43
9.0 %	925	605	203	194	153	119	102	77	61	46	39	41
9,5 %	874	573	161	185	146	113	85	66	59	42	38	39
10.0 %	621	543	154	147	140	109	82	64	51	38	37	38
10,5 %	589	516	147	140	111	104	79	61	49	37	34	35
11.0 %	559	368	141	134	107	100	76	52	48	36	31	30
11,5 %	532	351	135	129	103	80	63	50	41	32	28	31
12.0 %	507	335	130	124	99	77	61	49	40	32	30	28
12,5 %	484	320	100	95	95	74	59	47	39	31	27	29
13.0 %	463	306	96	92	91	72	57	46	38	28	29	26
13,5 %	443	293	92	88	71	69	55	38	37	30	24	27
14.0 %	425	281	89	85	68	67	44	37	31	27	26	24
14,5 %	407	270	86	82	66	65	43	36	31	24	22	28
15.0 %	261	260	83	79	64	50	42	35	30	21	23	21

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
15,5 %	251	250	80	77	62	49	40	34	29	23	25	25
16.0 %	242	241	77	74	60	47	39	33	24	23	21	22
16,5 %	233	155	75	72	58	46	38	27	28	20	18	23
17.0 %	224	149	55	70	56	44	37	26	23	22	22	19
17,5 %	216	144	53	51	54	43	36	31	27	17	20	24
18.0 %	209	139	51	49	53	42	28	25	22	19	18	20
18,5 %	202	135	50	48	39	41	28	24	22	19	16	20
19.0 %	195	130	48	46	37	40	27	24	18	16	16	21
19,5 %	189	126	47	45	36	39	26	23	21	16	19	21
20.0 %	183	122	46	44	35	38	26	23	21	18	17	17
20,5 %	177	118	44	43	35	37	25	22	17	18	17	17
21.0 %	172	115	43	41	34	27	24	22	20	14	15	18
21,5 %	167	112	42	40	33	26	24	17	16	13	15	18
22.0 %	162	108	41	39	32	26	23	21	16	15	13	19
22,5 %	157	105	40	38	31	25	23	21	16	15	13	19
23.0 %	153	102	39	37	30	24	22	16	15	15	13	14
23,5 %	148	99	38	36	30	24	22	16	15	15	16	14
24.0 %	144	97	37	36	29	23	16	16	15	13	11	15

Corporates, Sovereigns, and Banks; SMEs (5 Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving Retail; Other Retail

This definition of a critical number can be justified due to the development of the IRB-capital formula in Basel II: when the granularity adjustment (of Basel II) was cancelled, simultaneously the confidence level was increased from 0.995 to 0.999.[41] Thus, the reduction of the capital requirement by neglecting granularity was roughly compensated by an increase of the target confidence level. The risk of portfolios with a high number of credits will therefore be overestimated, if we assume that the actual target confidence level is 0.995, whereas the risk for a low number of credits will be underestimated. Thus, a critical number $\mathrm{I}_{c,abs}^{(fg)}$ of credits in the bucket exists, so that in each portfolio with a higher number of credits than $\mathrm{I}_{c,abs}^{(fg)}$ the VaR can be stated to be overestimated.

The critical numbers $\mathrm{I}_{c,per}^{(fg)}$ and $\mathrm{I}_{c,abs}^{(fg)}$ for homogeneous portfolios with different parameterizations of ϱ and PD are reported in Table 1 and Table 2. We do not only report the critical numbers under Basel II conditions, but also a wide range of parameter settings that might be relevant, if banks internal data are used for estimating ϱ. Due to the supervisory formula, this parameter is a function of PD for Corporates, Sovereigns, and Banks as well as for Small and Medium Enterprises (SMEs) and (other) retail exposures and remains fixed for residential mortgage exposures and revolving retail exposures.[42]

With definition type (18) the critical numbers $\mathrm{I}_{c,per}^{(fg)}$ vary from 23 to 35,986 credits (see Table 1), dependent on the probability of default PD and the correlation factor ϱ. In buckets with small probabilities of default as well as low correlation factors the idiosyncratic risk is relatively high, so that the portfolio must be substantially bigger to meet the goal. This means that in the worst case a portfolio must consist of at least 35,986 creditors to meet the assumptions of the ASRF framework at an accuracy of 5%. The same tendency can also be found for the target tolerance specification (19). We get critical numbers $\mathrm{I}_{c,abs}^{(fg)}$ ranging from 11 to 5,499 creditors (see Table 2), that are substantially lower compared to

[41] These were the major changes of the IRB-formula from the second to the third consultative document, see *Basel Committee On Banking Supervision* (2001, 2003).

[42] See *Basel Committee On Banking Supervision* (2004) paragraphs 272, 273, and 328 to 330. In both tables (rounded) parameters ϱ due to Basel II are marked. If one aims to measure ϱ from default series, one may refer to *Gordy/Heitfield* (2002), *Gordy* (2000) or *Düllmann/Trapp* (2004/2005). *Lopez* (2004) uses a KMV methodology. The results for estimating ϱ from portfolio data may differ from the correlations given in Basel II, see e.g. *Duellmann/Scheule* (2003) or *Dietsch/Petey* (2003), but overall the parameters given in Basel II are reasonable, see especially *Lopez* (2004).

the critical numbers of the target tolerance. Thus, the critical number $\mathrm{I}_{c,abs}^{(fg)}$ is less conservative. This is caused by the effect, that an increase of the confidence level for VaR calculations has a high impact especially on risk buckets with low default rates.

However, since for all those obligors still the ASRF assumptions (see section I) have to be valid, such big risk buckets may only be relevant for retail exposures in practice. Furthermore, it should be mentioned that these portfolio sizes are valid only for homogeneous portfolios. For heterogeneous portfolios these numbers can be considerably higher especially because the exposure weights differ between the obligors and thus concentration risk will occur.[43] Thus, an improvement of measuring the portfolio-VaR is indeed advisable. However, it has to be mentioned, that for portfolios with debtors incorporating low creditworthiness the ASRF solution is already sufficient for some hundred credits (or even less).

3. *Probing First Order Granularity Adjustment*

After auditing the adequacy of the ASRF solution (b) compared to the discrete, "true" solution (a) in context of a homogeneous risk bucket, we now investigate the accuracy of the first order granularity adjustment (solution (c)). Similar to section IV.2 we compare its accuracy with the discrete solution (a) but we additionally relate its result to the ASRF solution (b).

For the first (conservative) number $\mathrm{I}_{c,per}^{(1.\,\mathrm{Order\,Adj.})}$ we compare the analytically derived VaR including first order approximation (solution (c)) with the "true" VaR of the discrete, binomial solution (a) both on a 0.999 confidence level. Again, we aim to meet a target tolerance of β_T and we get

$$\mathrm{I}_{c,per}^{(1.\,\mathrm{Order\,Adj.})} = \inf\left(J : \left|\frac{\mathrm{VaR}_{0.999}^{(1.\,\mathrm{Order\,Adj.})}(\tilde{D}_T)}{\mathrm{VaR}_{0.999}^{(cg)}(\tilde{D}_T = \tilde{K}_T/n)} - 1\right| < \beta_T \text{ for all } n \in \mathbf{N}^{\geq J}\right) \quad (20)$$
$$\text{with} \quad \beta_{\mathrm{T}} = 0.05.$$

Thus, any analytically derived VaR of a risk bucket including more credits than $\mathrm{I}_{c,per}^{(1.\,\mathrm{Order\,Adj.})}$ does not differ from the "true" numerical derived VaR by more than 5 %.

The results for $\mathrm{I}_{c,per}^{(1.\,\mathrm{Order\,Adj.})}$ for homogeneous risk buckets with a specific (PD, ϱ)-combination are reported in Table 3 (see p. 102). Obviously, the

[43] We will come to that in section IV.5.

Table 3

Critical Number of Credits from that the First Order Adjustment can be Stated to be Sufficient for Measuring the True VaR (see Formula (20))

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
3.0 %	6100	4227	879	833	693	519	337	228	152	89	63	42
3,5 %	5517	3491	810	768	590	443	291	199	133	67	54	32
4.0 %	5027	3192	688	653	503	413	251	174	127	60	49	28
4,5 %	4169	2936	641	609	470	355	237	165	112	54	38	24
5.0 %	3846	2456	546	519	401	334	205	132	107	45	37	22
5,5 %	3564	2283	513	488	378	287	195	138	94	51	35	20
6.0 %	3317	2129	484	460	358	272	169	121	83	46	33	20
6,5 %	3098	1993	413	435	339	258	177	105	80	34	28	18
7.0 %	2902	1872	392	373	322	246	154	111	77	40	29	18
7,5 %	2450	1762	373	354	277	235	133	97	61	29	27	13
8.0 %	2309	1494	355	338	264	203	128	84	59	35	25	16
8,5 %	2181	1414	338	322	253	215	136	81	57	31	21	16
9.0 %	2065	1341	323	308	242	186	118	79	55	23	23	16
9,5 %	1958	1274	309	295	232	179	114	76	54	30	19	14
10.0 %	1861	1212	266	253	199	172	110	74	58	22	20	14
10,5 %	1771	1156	255	271	214	148	106	64	51	19	15	11
11.0 %	1689	1103	245	234	206	143	92	62	44	23	15	11
11,5 %	1612	1055	263	225	178	154	89	60	43	21	17	11
12.0 %	1541	1010	227	217	171	133	86	52	51	18	19	11
12,5 %	1476	968	219	209	166	129	74	57	46	19	23	11
13.0 %	1414	928	211	202	160	125	81	49	40	15	12	12
13,5 %	1357	892	204	195	155	121	88	54	30	16	10	8
14.0 %	1303	858	197	188	167	117	68	41	34	17	8	8
14,5 %	1253	825	191	182	145	101	66	45	33	12	8	8
15.0 %	1206	795	185	176	141	110	64	56	28	14	15	8

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
15,5 %	1162	767	179	171	121	107	62	49	36	14	13	12
16.0 %	1120	740	154	166	118	104	69	37	31	16	13	9
16,5 %	1081	714	168	161	114	101	67	51	23	16	11	9
17.0 %	1044	690	145	156	125	87	58	35	30	9	11	9
17,5 %	1009	668	159	152	108	96	49	30	22	7	11	9
18.0 %	976	646	154	131	105	83	55	39	18	7	9	9
18,5 %	944	626	150	128	115	91	61	43	25	7	9	9
19.0 %	914	606	146	124	112	79	53	28	21	13	9	9
19,5 %	886	588	142	136	97	77	45	32	17	18	9	9
20.0 %	859	570	123	118	95	75	44	36	20	14	9	9
20,5 %	834	554	120	129	104	73	43	35	13	12	7	9
21.0 %	809	538	117	112	90	63	42	30	16	10	7	9
21,5 %	786	523	128	109	99	70	41	25	19	10	7	9
22.0 %	764	508	111	106	86	77	51	29	22	8	7	9
22,5 %	743	494	108	104	84	67	40	20	14	8	7	9
23.0 %	722	481	119	114	92	57	39	36	11	8	7	9
23,5 %	703	468	116	99	90	72	38	24	27	8	7	9
24.0 %	684	456	101	97	88	55	32	16	18	8	7	9

Corporates, Sovereigns, and Banks · SMEs (5 Mio. < Sales < 50 Mio.) · SMEs (Sales < 5 Mio.) · Mortgage · Revolving Retail · Other Retail

Table 4

Critical Number of Credits from that the First Order Adjustment on Confidence Level 0.995 Exceeds the Infinite Fine Granularity on Confidence Level 0.999 (see Formula (21))

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
3.0 %	5170	3544	973	935	769	626	441	327	255	164	146	128
3,5 %	4029	2773	774	744	615	501	356	265	209	136	122	109
4.0 %	3231	2232	633	609	504	413	295	221	175	116	105	95
4,5 %	2650	1836	528	508	422	347	249	188	150	101	91	85
5.0 %	2213	1538	448	431	359	296	214	162	130	89	81	76
5,5 %	1875	1307	385	371	310	256	186	142	114	79	72	69
6.0 %	1609	1124	335	323	270	224	163	125	101	71	65	63
6,5 %	1395	977	295	284	238	198	145	112	91	64	60	59
7.0 %	1220	856	261	252	211	176	130	100	82	59	55	55
7,5 %	1075	757	233	225	189	158	117	91	74	54	50	51
8.0 %	955	673	209	202	170	142	106	83	68	50	47	48
8,5 %	853	602	189	182	154	129	96	75	62	46	44	45
9.0 %	766	542	171	165	140	117	88	69	58	43	41	43
9,5 %	691	490	156	151	128	108	81	64	53	40	38	41
10.0 %	626	445	143	138	117	99	75	59	50	38	36	39
10,5 %	570	405	131	127	108	91	69	55	46	36	34	37
11.0 %	521	371	121	117	100	84	64	51	43	34	32	36
11,5 %	477	340	112	108	92	78	60	48	40	32	31	34
12.0 %	439	313	104	100	86	73	56	45	38	30	29	33
12,5 %	404	289	96	93	80	68	52	42	36	29	28	32
13.0 %	374	268	90	87	74	63	49	40	34	27	27	31
13,5 %	346	248	84	81	70	59	46	37	32	26	26	30
14.0 %	322	231	78	76	65	56	43	35	30	25	24	29
14,5 %	299	215	74	71	61	52	41	33	29	24	24	28
15.0 %	279	201	69	67	58	49	39	32	27	23	23	28

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
15,5 %	261	188	65	63	54	47	36	30	26	22	22	27
16.0 %	244	176	61	59	51	44	35	29	25	21	21	26
16,5 %	229	165	58	56	48	42	33	27	24	20	20	26
17.0 %	215	155	55	53	46	40	31	26	23	20	20	25
17,5 %	202	146	52	50	43	38	30	25	22	19	19	25
18.0 %	190	138	49	48	41	36	28	24	21	18	18	24
18,5 %	180	130	46	45	39	34	27	23	20	18	18	24
19.0 %	170	123	44	43	37	32	26	22	19	17	17	23
19,5 %	160	116	42	41	36	31	25	21	19	17	17	23
20.0 %	152	110	40	39	34	29	24	20	18	16	16	22
20,5 %	144	105	38	37	32	28	23	19	17	16	16	22
21.0 %	136	99	36	35	31	27	22	18	17	15	16	22
21,5 %	129	94	35	34	29	26	21	18	16	15	15	22
22.0 %	123	90	33	32	28	25	20	17	15	14	15	21
22,5 %	117	85	32	31	27	24	19	17	15	14	15	21
23.0 %	111	81	30	29	26	23	18	16	14	14	14	21
23,5 %	106	78	29	28	25	22	18	15	14	13	14	21
24.0 %	101	74	28	27	24	21	17	15	14	13	14	20

Corporates, Sovereigns, and Banks — SMEs (5 Mio. < Sales < 50 Mio.) — SMEs (Sales < 5 Mio.) — Mortgage — Revolving Retail — Other Retail critical number varies from 7 to 6,100 credits. Compared to the ASRF solution (see Table 1 in section IV.2), the critical values drop by 83.04 % at a stretch. Precisely, we find that the number of credits that is necessary to ensure a good approximation of the "true" VaR is significantly lower with the adjustment (c) than without the adjustment (b). For example, a high quality retail portfolio (AAA) must consist of 5,027 compared to 26,051 credits if we neglect the first order adjustment. A medium quality corporate portfolio (BBB) must contain 106 compared to 442 credits. Thus, the minimum portfolio size should be small enough to hold for real world portfolios and we may come to the conclusion, that the first order adjustment works fine even with our conservative definition of a critical value.

Thus, we are able to use the ASRF formula with the first order granularity adjustment (c) as a (still progressive biased) proxy for the discrete numerical solution (a) and we are able to relate it to the ASRF formula (b). We do that by defining a critical value $\mathrm{I}_{c,\,abs}^{(1.\,\mathrm{Order\,Adj.})}$ of credits similar to the definition (19), but this time we proclaim, that the VaR of the ASRF solution without first order granularity adjustment (b) at confidence level of 0.999 should not exceed the VaR with first order granularity adjustment (c) at confidence level of 0.995. We write

$$\text{(21)} \qquad \mathrm{I}_{c,\,abs}^{(1.\,\mathrm{Order\,Adj.})} = \sup\left(J : \mathrm{VaR}_{0.999}^{(fg)}(\tilde{D}_T) < \mathrm{VaR}_{0.995}^{(1.\,\mathrm{Order\,Adj.})}\left(\tilde{D}_T = \frac{\tilde{K}_T}{J}\right)\right).$$

Consequently, the confidence level is increased by a buffer of 4 basis points, which should incorporate the idiosyncratic risk approximated by the first order granularity adjustment.

The critical numbers of credits $\mathrm{I}_{c,\,abs}^{(1.\,\mathrm{Order\,Adj.})}$ are shown in Table 4 (see p. 104). They contain a range from 14 to 5,170. It is interesting to note, that these critical values do not differ widely from the numbers $\mathrm{I}_{c,\,abs}^{(1.\,\mathrm{Order\,Adj.})}$, where we compared the VaR of the ASRF solution (b) with the "true" VaR using the numerical, time-consuming discrete formula. Precisely, the average percentage difference between the critical numbers of Table 2 and Table 4 is less than 10 %. That means that the diversification behaviour of the coarse grained solution and the first order approximation is very similar, i.e. the first order adjustment is a good approximation of the idiosyncratic risk of coarse grained portfolios.

4. Probing Second Order Granularity Adjustment

Finally, we would like to test the approximation if the (first and) second order adjustment is added to the ASRF formula and we get the solution (d). Similar to section IV.2 and IV.3, we firstly examine the VaR according to this new formula (d) in comparison to the "exact" VaR from the coarse grained solution (a). Additionally, we analyse its performance with respect to the ASRF solution.

Again, we calculate a critical number $\mathrm{I}_{c,per}^{(1.+2.\,\mathrm{Order\,Adj.})}$ of credits to test the approximation accuracy with reference to the coarse grained formula (a) according to the "percentage" accuracy with a target tolerance of 5% by

$$\text{(22)} \qquad \mathrm{I}_{c,per}^{(1.+2.\,\mathrm{Order\,Adj.})} = \inf\left(J : \left| \frac{\mathrm{VaR}_{0.999}^{(1.+2.\,\mathrm{Order\,Adj.})}(\tilde{D}_T)}{\mathrm{VaR}_{0.999}^{(cg)}(\tilde{D}_T = \tilde{K}_T/n)} - 1 \right| < \beta_T \text{ for all } n \in \mathbf{N}^{\geq J} \right)$$
$$\text{with} \quad \beta_T = 0.05,$$

using the (first and) second order adjustment as an approximation of the coarse grained portfolio.

The results are presented in Table 5. Now, the critical number of credits ranges from 17 to 10,993. Compared to the ASRF solution (a), see Table 1 in section IV.2, the necessary number of credits to meet the requirements can be reduced to 33.5 percent on average. Thus, the second order adjustment is capable to detect idiosyncratic risk caused by a finite number of debtors to certain extend. However, if we compare the result with the ones of the only use of the first order adjustment (see Table 3 in section IV.3), the second order adjustment performs less. This might be due to the fact that the confidence level of 0.999 is very conservative and thus the more conservative first order adjustment (c) works better than the second order adjustment (d).

We are able to verify this result by analysing the second order adjustment (d) in comparison with the exact ASRF solution (a). Therefore we introduce a critical number $\mathrm{I}_{c,abs}^{(1.+2.\,\mathrm{Order\,Adj.})}$ of credits, similar to the definition (22) in section IV.3. We get

$$\text{(23)} \qquad \mathrm{I}_{c,abs}^{(1.+2.\,\mathrm{Order\,Adj.})} = \sup\left(J : \mathrm{VaR}_{0.999}^{(fg)}(\tilde{D}_T) < \mathrm{VaR}_{0.995}^{(1.+2.\,\mathrm{Order\,Adj.})}\left(\tilde{D}_T = \frac{\tilde{K}_T}{J} \right) \right).$$

Table 5

Critical Number of Credits from that the First plus Second Order Adjustment can be Stated to be Sufficient for Measuring the True VaR (see Formula (22))

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
3.0 %	10993	7338	1796	1770	1417	1107	746	522	392	222	185	130
3,5 %	9309	6251	1503	1427	1150	941	620	440	327	193	163	115
4.0 %	7494	5077	1260	1252	1014	802	534	384	280	167	140	103
4,5 %	6405	4367	1109	1054	858	683	460	323	255	148	120	90
5.0 %	5864	3768	979	930	761	609	414	293	225	127	115	83
5,5 %	5056	3256	866	824	677	544	373	266	199	118	103	78
6.0 %	4362	3021	767	730	603	486	321	242	182	107	94	70
6,5 %	4055	2622	680	647	537	435	304	210	167	100	86	64
7.0 %	3509	2452	641	610	478	390	260	191	147	90	76	63
7,5 %	3286	2132	570	542	453	349	248	183	141	84	74	60
8.0 %	2844	2006	505	481	404	332	237	158	123	79	67	55
8,5 %	2679	1892	480	457	385	297	214	160	119	71	63	51
9.0 %	2529	1649	457	406	343	284	193	146	109	69	57	49
9,5 %	2394	1563	406	387	328	254	174	133	105	67	58	51
10.0 %	2077	1484	388	370	292	243	168	128	91	60	50	42
10,5 %	1974	1412	344	354	280	234	161	116	88	56	49	43
11.0 %	1879	1231	330	314	269	209	145	106	81	52	48	41
11,5 %	1791	1175	316	302	239	201	140	109	88	51	45	38
12.0 %	1710	1123	304	290	230	194	126	99	76	52	41	39
12,5 %	1484	1075	269	257	222	173	131	96	74	51	42	37
13.0 %	1421	1030	259	248	214	167	127	87	63	43	43	34
13,5 %	1362	897	250	239	190	149	106	79	70	42	37	34
14.0 %	1307	861	241	230	184	144	111	76	64	39	38	31
14,5 %	1256	828	233	203	177	139	92	80	54	38	34	32
15.0 %	1208	797	206	197	172	135	97	67	61	33	35	28

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
15,5 %	1163	768	199	190	152	131	94	65	52	39	31	29
16.0 %	1120	741	193	184	147	127	84	74	51	34	34	30
16,5 %	1081	715	187	178	143	113	89	67	46	38	30	26
17.0 %	938	690	181	173	152	120	73	56	45	33	28	26
17,5 %	906	600	176	168	135	106	71	64	51	31	26	27
18.0 %	876	646	155	163	131	103	69	58	43	32	24	28
18,5 %	847	562	150	144	115	101	74	52	42	30	27	23
19.0 %	820	544	146	140	124	98	72	51	41	26	25	23
19,5 %	795	527	142	150	109	86	64	45	37	29	23	24
20.0 %	770	511	138	132	106	93	57	44	33	27	26	25
20,5 %	747	496	134	115	93	91	67	43	42	23	21	26
21.0 %	725	482	131	125	101	80	60	39	38	21	24	26
21,5 %	704	468	114	122	88	78	53	42	31	24	22	20
22.0 %	684	455	124	119	96	68	57	41	34	22	22	20
22,5 %	665	442	121	116	94	67	56	44	39	22	20	21
23.0 %	647	430	106	101	82	73	44	32	30	20	17	22
23,5 %	629	419	103	99	80	64	43	35	24	18	21	22
24.0 %	613	408	101	108	78	62	43	38	29	21	18	23

Corporates, Sovereigns, and Banks | SMEs (5 Mio. < Sales < 50 Mio.) | SMEs (Sales < 5 Mio.) | Mortgage | Revolving Retail | Other Retail

Table 6

Critical Number of Credits from that the First plus Second Order Adjustment on Confidence Level 0.995 Exceeds the Infinite Fine Granularity on Confidence Level 0.999 (see Formula (23))

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
3.0 %	4285	2942	810	778	640	521	367	272	214	140	125	114
3,5 %	3266	2254	633	609	503	411	292	218	173	115	104	97
4.0 %	2560	1776	508	489	406	333	238	180	143	97	89	84
4,5 %	2050	1429	417	401	334	275	198	151	121	83	77	75
5.0 %	1671	1170	347	335	279	231	168	128	103	73	68	67
5,5 %	1380	971	294	283	237	196	144	111	90	64	60	61
6.0 %	1153	815	251	242	203	169	124	96	79	57	54	56
6,5 %	973	691	216	209	176	147	109	85	70	52	49	51
7.0 %	827	590	188	182	153	128	96	75	62	47	44	48
7,5 %	708	507	164	159	135	113	85	67	56	43	41	44
8.0 %	610	439	145	140	119	100	76	60	50	39	38	42
8,5 %	527	382	128	124	106	89	68	54	46	36	35	39
9.0 %	458	333	114	110	94	80	61	49	42	33	32	37
9,5 %	399	292	102	98	84	72	55	45	38	31	30	35
10.0 %	349	257	91	88	76	65	50	41	35	29	28	33
10,5 %	306	226	82	79	68	59	46	37	32	27	27	32
11.0 %	268	200	74	72	62	53	42	34	30	25	25	31
11,5 %	264	177	67	65	56	48	38	32	28	24	24	29
12.0 %	271	156	60	59	51	44	35	29	26	22	22	28
12,5 %	266	173	55	53	46	40	32	27	24	21	21	27
13.0 %	257	172	50	48	42	37	30	25	22	20	20	26
13,5 %	248	167	45	44	39	34	27	23	21	19	19	25
14.0 %	238	162	41	40	36	31	25	22	20	18	18	24
14,5 %	229	156	38	37	33	29	24	20	18	17	18	24
15.0 %	219	150	34	34	30	26	22	19	17	16	17	23

	AAA up to AA- 0.03 %	A- up to A+ 0.05 %	BBB+ 0.32 %	BBB 0.34 %	BBB- 0.46 %	BB+ 0.64 %	BB 1.15 %	BB- 1.97 %	B+ 3.19 %	B 8.99 %	B- 13.01 %	CCC up to C 30.85 %
15,5 %	210	144	38	36	27	24	20	18	16	15	16	22
16.0 %	201	139	38	36	28	23	19	17	15	15	15	22
16,5 %	193	133	37	36	29	21	18	16	14	14	15	21
17.0 %	185	128	37	35	29	22	16	15	14	13	14	21
17,5 %	177	123	36	34	28	23	15	14	13	13	14	20
18.0 %	170	118	35	33	28	23	14	13	12	12	13	20
18,5 %	163	113	34	33	27	22	13	12	12	12	13	19
19.0 %	156	109	33	32	26	22	15	11	11	11	12	19
19,5 %	150	105	32	31	26	21	15	11	10	11	12	19
20.0 %	145	101	31	30	25	21	15	10	10	11	12	18
20,5 %	139	97	30	29	24	20	15	10	9	10	11	18
21.0 %	134	94	29	28	24	20	14	9	9	10	11	18
21,5 %	129	90	28	27	23	19	14	10	8	10	11	17
22.0 %	124	87	27	26	22	19	14	10	8	9	10	17
22,5 %	120	84	26	26	22	18	14	10	8	9	10	17
23.0 %	115	81	26	25	21	18	13	10	7	9	10	16
23,5 %	111	78	25	24	20	17	13	10	7	8	9	16
24.0 %	108	75	24	23	20	17	13	10	7	8	9	16

Corporates, Sovereigns, and Banks — SMEs (5 Mio. < Sales < 50 Mio.) — SMEs (Sales < 5 Mio.) — Mortgage — Revolving Retail — Other Retail

So for each risk bucket with at least $I_{c,abs}^{(1.+2.\,\text{Order Adj.})}$ number of credits the idiosyncratic risk, measured by the second order adjustment on a confidence level 0.995, is included in the confidence level premium of 4 basis points of the ASRF solution (on a confidence level 0.999).

These critical numbers presented in Table 6 (see p. 110) range from 7 to 4,285. Obviously, these results are considerably higher than those of Table 4 and therefore the predefined target value of accuracy is reached with lower numbers of credits. Thus, the idiosyncratic risk is underestimated with the second order adjustment compared to the numerical "true" solution (a) (see the results in section IV.2) and is not measured with such a high accuracy as the first order adjustment does (see section IV.3). Concretely, this value is reduced by averaged 32.7 percent credits.

To conclude, the second order adjustment (d) converges faster to the asymptotic value of the ASRF solution (b), which confirms the findings of section IV.1. A possible reason is that the VaR measure using the first order approximation may be "corrected" into the direction of the ASRF solution by incorporating the second order adjustment. The possibility of this behaviour is given due to the alternating sign in the derivatives of VaR, see formula (12).[44] Thus, taking more derivatives into account could solve the problem, but would lead to even more uncomfortable equations.[45] Despite these theoretical questions, it can be stated that in homogeneous portfolios an excellent approximation of the true VaR can be achieved with the granularity adjustment.

5. *Probing Granularity for Inhomogeneous Portfolios*

The previous analyses showed that the granularity adjustment works fine for homogeneous portfolios. In this section we test if the approximation accuracy of the presented general formulas will hold for portfolios consisting of loans with different exposures and credit qualities. This means, that the credits in the portfolio vary in the exposure weight and in the probability of default, and we analyse if the gross loss rate for coarse grained portfolios could still be quantified satisfactory by the granularity adjustment.

[44] This is true not only for the first five derivatives but also for all following derivatives; see the general formula for all derivatives of VaR in *Wilde* (2003).

[45] However, we also have to take into consideration that the Taylor series is potentially not convergent at all or does not converge to the correct value. For a further discussion see *Martin/Wilde* (2002) and *Wilde* (2003).

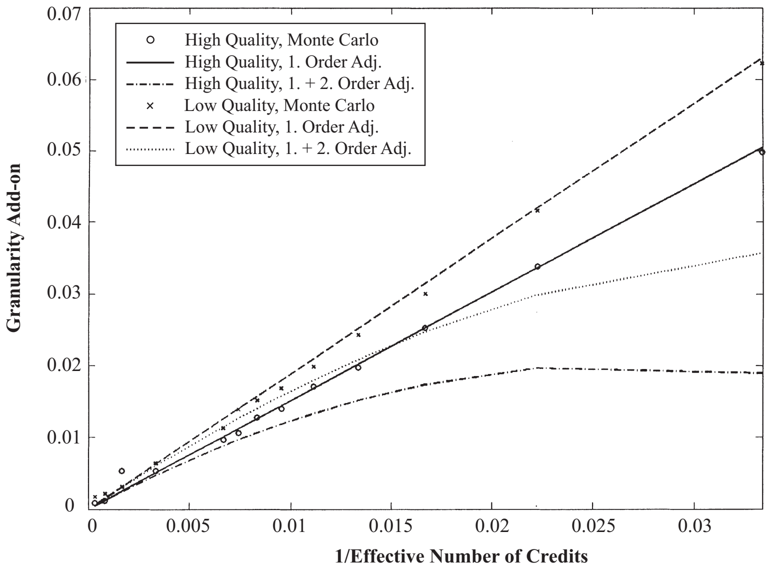

Figure 3: Granularity Add-on for Heterogeneous Portfolios Calculated Analytically with First Order (Solid Lines) and Second Order (Dotted Lines) Adjustments as well as with Monte Carlo Simulations (x and o) Using 3 Million Trials

Concretely, we examine high quality portfolios with probabilities of default ranging from 0.02 % to 0.79 % and lower quality portfolios with probabilities of default ranging from 0.2 % to 7.9 %. Additionally, we define a basic risk bucket consisting of 20 loans with exposures between 35 and 200 million €.[46] In order to measure the portfolio size with respect to concentration risk we use the effective number of loans

(24) $$J^* := 1 \Big/ \sum_{j=1}^{J} w_j^2$$

rather than the number of loans J.[47] Consequently, this effective number is more than 25 % below the true number of credits.

[46] The used portfolio is based on *Overbeck* (2000), see also *Overbeck/Stahl* (2003), but reduced to 20 loans to achieve more test portfolios.

[47] The effective number J^* of credits is based on the *Herfindahl-Hirshman* index $H := 1/J^*$, that is preferably used as a measure of concentration in credit portfolios, see *Gordy* (2003) and *Basel Committee On Banking Supervision* (2001b), paragraphs 432 and 434.

A variation of portfolio size is reached by reproducing the basic risk bucket so that portfolios with 40, 60, …, 400, 800, 1600 and 4000 loans result. Using an asset correlation $\varrho = 20\%$ and confidence level of 0.999, we compute the granularity add-on with the presented first order and second order adjustment.[48] Because the exact value can not be determined analytically for heterogeneous portfolios, we compute the "true" VaR with Monte Carlo simulations using 3 million trials.[49] Finally, we compare this "true" VaR with the ASRF solution, so that we receive the granularity add-on.

The simulated results for granularity add-on for the high quality portfolios and low quality portfolios are presented in Figure 3 (see the circles and dots). Therefore, the add-on for the minimum size of 40 loans with $1/J \approx 0.035$ is 5.0 % (6.2 %) for the high (low) quality portfolio. This is equal to a relative correction of $+112.5\%$ $(+30.5\%)$ compared to a hypothetical infinitely fine grained portfolio. This shows again the relative high impact of idiosyncratic risks in small high quality portfolios. With shifting to bigger sized portfolios the effective number of credits shifts to zero and the granularity add-on decreases almost exactly linear in terms of $1/J^*$ – even for high quality portfolios. This result is contrary to *Gordy* (2003), who exhibits a concave characteristic of the granularity add-on. This might be due to the fact, that *Gordy* (2003) uses a CreditRisk⁺ framework, whereas we analysed the effect of the granularity with the CreditMetrics one-factor model that is consistent with the Basel II assumptions.

Thus, the granularity add-on in Figure 3 can be approximated with a linear function. Indeed, the (linear) first order adjustment is a very good approximation for heterogeneous portfolios of high as well as low quality. Just like in the previous sections, the second order adjustment leads to a reduction of the granularity add-on, thus it can be characterized as less conservative, but comparing the results we strongly recommend the first order adjustment.

V. Conclusion

Presently discussed analytical solutions for risk quantification of credit portfolio models especially rely on the assumptions of an infinite number

[48] For the concrete formulas see Appendices A and C.

[49] As in *Gordy* (2003) we firstly used 300,000 Monte Carlo trials for calculation of the 0.99 confidence level (leading to 3,000 hits in the tail). However, on a 0.999 confidence level the VaRs were not stable and thus we recommend 3 million trials (also with 3,000 hits in the tail) that seemed to be appropriate in our case.

of credits and of only one systematic factor. Thus, those analytical frameworks do not account for "single name" and "sector" concentration risks. This problem is discussed intensively by the financial authorities and it is especially considered in Pillar 2 of Basel II. Since one could get "sector" concentration risk under control by building small risk buckets, the "single name" concentration risk and the infinite granularity assumption, respectively, might be the most critical assumption. To cope with this problem, recently an add-on factor was developed, that adjusts the analytical solution for portfolios of finite size and therefore might serve as a simple solution for quantifying "single name" concentration risk under Pillar 2. In this article we briefly reviewed the general framework of this (first order) granularity adjustment for medium sized risk buckets. Furthermore, we derived an additional (second order) adjustment for small risk buckets, since an improvement due to the higher order term is expected in the literature. We implemented this adjustment on the *Vasicek* model that also builds the basis of the Basel II credit risk formula. We carried out a detailed numerical study. In this study we reviewed the accuracy of the infinite granularity assumption for credit portfolios with a finite number of credits, as well as the improvement of accuracy with so-called first and second order granularity adjustments. We received some critical values for the minimum numbers of credits for the analytical solutions compared to the numerical "exact" solutions under the risk measure Value at Risk (VaR). As far as we know, such a study was carried out for the first time. We came to the conclusion, that the critical number of credits for approving the assumption of infinite granularity is influenced by the probability of default, the asset correlation and of course the acquired accuracy of the analytical formula to great extent. The number of credits varies enormously, e.g. from 1,371 to 23,989 for a high-quality portfolio (A-rated) and from 23 to 205 for an extremely low-quality portfolio (CCC-rated). With the use of the first order granularity adjustment we could reduce these ranges drastically. The critical number of credits is in the bandwidth 456 to 4,227 (A-rated) and 9 to 42 (CCC-rated) and thus, the postulated accuracy should be obtained in many real-world portfolios. Additionally, the second order adjustment does not seem to work for a conservative risk measure like the VaR, since it reduces the add-on factor. To conclude, we think that in general the assumption of an infinitely fine grained portfolio seems to hold even for relatively small portfolios, especially if the first order granularity adjustment is incorporated.

Appendix A

With reference to *Emmer/Tasche* (2005),[50] and to *Pykhtin/Dev* (2002) for the homogenous case, the first order granularity adjustment for inhomogeneous portfolios is

$$
\begin{aligned}
\triangle l_1 = -\frac{1}{2}\Bigg[& q_{1-z}(\tilde{x}) \frac{\sum_{i=1}^{J} w_i^2 \left(N(y_i) - N^2(y_i)\right)}{\sum_{i=1}^{J} w_i \left(\sqrt{\varrho}/\sqrt{1-\varrho}\right) n(y_i)} \\
& + \frac{\sum_{i=1}^{J} w_i^2 \left(\frac{\sqrt{\varrho}}{\sqrt{1-\varrho}} n(y_i) - 2\,\frac{\sqrt{\varrho}}{\sqrt{1-\varrho}}\, N(y_i)\cdot n(y_i)\right)}{\sum_{i=1}^{J} w_i \left(\sqrt{\varrho}\Big/\sqrt{1-\varrho}\right) n(y_i)} \\
& + \frac{\sum_{i=1}^{J} \left[w_i^2 \left(N(y_i) - N^2(y_i)\right)\right] \cdot \sum_{i=1}^{J} w_i \left[\frac{\varrho}{1-\varrho} y_i \cdot n(y_i)\right]}{\left(\sum_{i=1}^{J} w_i \left(\sqrt{\varrho}\Big/\sqrt{1-\varrho}\right) n(y_i)\right)^2} \Bigg]_{y_i = \frac{N^{-1}(PD_i) - \sqrt{\varrho}\cdot q_{1-z}(\tilde{x})}{\sqrt{1-\varrho}}} .
\end{aligned}
\tag{A1}
$$

Appendix B

For any $m \in \mathbf{N}$ the $(m+1)^{\text{th}}$ element of the *Taylor* series can be written as

$$
\frac{\lambda^m}{m!}\left[\frac{\partial^m \mathrm{VaR}_\alpha(\tilde{Y} + \lambda\tilde{Z})}{\partial\lambda^m}\right]_{\lambda=0} = g \circ \left(\frac{\lambda^m}{m!} \sum_{p \prec m} \prod_{r=1}^{m} \left(\mu_r\left[\tilde{Z} \middle| \tilde{Y} = l\right]\right)^{e_{pr}}\right)\Bigg|_{l = \mathrm{VaR}_Z(\tilde{Y})} ,
\tag{A2}
$$

with the notation $p \prec m$ to indicate that p is a partition of m, e_i represents the frequency how often a number i appears in a partition p, and g is a function that is independent of the number of credits J. With μ_r as the r^{th} (conditional) moment about the origin and η_r as the r^{th} (conditional) moment about the mean it is possible to write

$$
\begin{aligned}
\lambda^m \prod_{r=1}^{m} \left(\mu_r\left[\tilde{Z} \middle| \tilde{Y} = l\right]\right)^{e_{pr}} &= \prod_{r=1}^{m} \left(\mu_r\left[\lambda\tilde{Z} \middle| \tilde{Y} = l\right]\right)^{e_{pr}} \\
&= \prod_{r=1}^{m} \left(\mu_r\left[\left(\tilde{D}_T \middle| \tilde{Y} = l\right) - E\left[\tilde{D}_T \middle| \tilde{Y} = l\right]\right]\right)^{e_{pr}} = \prod_{r=1}^{m} \left(\eta_r\left[\tilde{D}_T \middle| \tilde{Y} = l\right]\right)^{e_{pr}}
\end{aligned}
\tag{A3}
$$

[50] In addition to the formula for the granularity adjustment, the authors consider the contribution of single borrowers to entire portfolio risk via the partial derivative with respect to the exposure weight w_i, but this aspect is not the purpose of this article.

for each partition of m.[51] Due to the limitation $\tilde{I}_i \in [-1,1]\ \forall i \in \{1,\ldots,J\}$ there exists a finite constant η_r^*, so that under assumption of conditional independent defaults we have

$$\text{(A4)}\qquad \eta_r[\tilde{D}_T|\tilde{x}=x] = \eta_r\left[\sum_{i=1}^{J} w_i \cdot \tilde{I}_i \middle| \tilde{Y}=l\right] = \sum_{i=1}^{J} w_i^r \cdot \eta_r\left[\tilde{I}_i \middle| \tilde{Y}=l\right] = \eta_r^* \cdot \sum_{i=1}^{J} w_i^r.$$

Revisiting equations (A2) to (A4) it is straightforward to see that only for $m=3$ and $m=4$ there exist terms which are maximum of Order $O(1/J^2)$

$$\begin{aligned}
\sum_{p\prec 3}\prod_{r=1}^{3}\left(\eta_r\left[\tilde{D}_T\middle|\tilde{Y}=l\right]\right)^{e_{pr}} &= \eta_3\left[\tilde{D}_T\middle|\tilde{Y}=l\right] = \eta_3^*\cdot\sum_{i=1}^{J} w_i^3 \le \eta_3^*\cdot\sum_{i=1}^{J}\left(\frac{b}{J\cdot a}\right)^3 \\
&= \eta_3^*\cdot\left(\frac{b}{a}\right)^3\cdot\frac{1}{J^2} = O\left(\frac{1}{J^2}\right),
\end{aligned}$$

$$\begin{aligned}
\text{(A5)}\quad \sum_{p\prec 4}\prod_{r=1}^{4}\left(\eta_r\left[\tilde{D}_T\middle|\tilde{Y}=l\right]\right)^{e_{pr}} &= \eta_4\left[\tilde{D}_T\middle|\tilde{Y}=l\right] + \left(\eta_2\left[\tilde{D}_T\middle|\tilde{Y}=l\right]\right)^2 \\
&= \eta_4^*\cdot\sum_{i=1}^{J} w_i^4 + \left(\eta_2^*\cdot\sum_{i=1}^{J} w_i^2\right)^2 \\
&\le \eta_4^*\cdot\sum_{i=1}^{J}\left(\frac{b}{J\cdot a}\right)^4 + \left(\eta_2^*\cdot\sum_{i=1}^{J}\left(\frac{b}{J\cdot a}\right)^2\right)^2 \\
&= \eta_4^*\cdot\left(\frac{b}{a}\right)^4\cdot\frac{1}{J^3} + \left(\eta_2^*\cdot\left(\frac{b}{a}\right)^2\cdot\frac{1}{J}\right)^2 = O\left(\frac{1}{J^3}\right) + O\left(\frac{1}{J^2}\right),
\end{aligned}$$

with $a \le E_i \le b$ for some $0 < a \le b$ and all i. All terms of higher derivatives of VaR are at least of Order $O(1/J^3)$.

Appendix C

In order to shorten the equation (16) we set $\mu_1 := \mu_1(x)$, $\eta_{2,3} := \eta_{2,3}(x)$, $n_x := n(x)$, and we get the following general form of the second order adjustment

$$\begin{aligned}
\triangle l_2 &= \left[\frac{1}{6\,n_x}\frac{d}{dx}\left(\frac{1}{d\mu_1/dx}\frac{d}{dx}\left[\frac{\eta_3\,n_x}{d\mu_1/dx}\right]\right)\right. \\
\text{(A6)}\qquad &\qquad \left. + \frac{1}{8\,n_x}\frac{d}{dx}\left(\frac{1}{n_x}\frac{1}{d\mu_1/dx}\left[\frac{d}{dx}\left(\frac{\eta_2\,n_x}{d\mu_1/dx}\right)\right]^2\right)\right]_{x=q_{1-z}(\tilde{x})} \\
&=: \left[\triangle l_{2,1} + \triangle l_{2,2}\right]_{x=q_{1-z}(\tilde{x})}.
\end{aligned}$$

---

[51] To illustrate that this will indeed hold for each partition, we demonstrate an example for $m=5 : \lambda \sum_{p\prec 5}\prod_{r=1}^{5}(\mu_r)^{e_{pr}} = \lambda\Big(\mu_5 + \mu_4\cdot\mu_1 + \mu_3\cdot(\mu_1)^2 + \mu_3\cdot\mu_2 + \mu_2\cdot(\mu_1)^3 + (\mu_2)^2\cdot\mu_1 + (\mu_1)^5\Big)$.

First, the term $\triangle l_{2,1}$ will be examined

$$\text{(A7)}\quad \begin{aligned}\triangle l_{2,1} =&\frac{1}{6}\left[\frac{d}{dx}\left(\frac{1}{d\mu_1/dx}\right)\left(\underbrace{\frac{1}{n_x}\frac{d}{dx}\left(\eta_3\, n_x\right)}_{\mathrm{I}}\frac{1}{d\mu_1/dx}+\eta_3\frac{d}{dx}\left(\frac{1}{d\mu_1/dx}\right)\right)\right.\\ &\left.+\frac{1}{d\mu_1/dx}\frac{1}{n_x}\frac{d}{dx}\left[\underbrace{\frac{d}{dx}\left(\eta_3\, n_x\right)\frac{1}{d\mu_1/dx}}_{\mathrm{II}}+\underbrace{\eta_3\, n_x\frac{d}{dx}\left(\frac{1}{d\mu_1/dx}\right)}_{\mathrm{III}}\right]\right].\end{aligned}$$

During the derivation, there will be use of following expressions

$$\text{(A8)}\quad \begin{aligned}&n_x=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\frac{dn_x}{dx}=-x\cdot n_x,\frac{d^2n_x}{dx^2}=(x^2-1)n_x \quad \text{and}\\ &\frac{d}{dx}\left(\frac{1}{d\mu_1/dx}\right)=-\frac{d^2\mu_1/dx^2}{\left(d\mu_1/dx\right)^2}.\end{aligned}$$

Then, (I) can be transformed into

$$\text{(A9)}\quad \frac{1}{n_x}\frac{d}{dx}\left(\eta_3\cdot n_x\right)=\frac{d\eta_3}{dx}-\eta_3\cdot x.$$

The derivative of (II) can be calculated as follows

$$(A10)\quad \begin{aligned}\frac{d}{dx}\left(\frac{d}{dx}\left(\eta_3\cdot n_x\right)\frac{1}{d\mu_1/dx}\right)=&\left(\frac{d^2\eta_3}{dx^2}n_x+2\frac{d\eta_3}{dx}\frac{dn_x}{dx}+\eta_3\frac{d^2n_x}{dx^2}\right)\frac{1}{d\mu_1/dx}\\ &-\left(\frac{d\eta_3}{dx}n_x+\eta_3\frac{dn_x}{dx}\right)\frac{d^2\mu_1/dx^2}{\left(d\mu_1/dx\right)^2}.\end{aligned}$$

Taking the derivative of (III) results in

$$\text{(A11)}\quad \begin{aligned}\frac{d}{dx}\left(\eta_3\cdot n_x\left(-\frac{d^2\mu_1/dx^2}{\left(d\mu_1/dx\right)^2}\right)\right)=&\left(-\frac{d\eta_3}{dx}n_x-\eta_3\frac{dn_x}{dx}\right)\frac{d^2\mu_1/dx^2}{\left(d\mu_1/dx\right)^2}\\ &-\eta_3\cdot n_x\left(\frac{\left(d\mu_1/dx\right)^2\left(d^3\mu_1/dx^3\right)-2\left(d\mu_1/dx\right)\left(d^2\mu_1/dx^2\right)^2}{\left(d\mu_1/dx\right)^4}\right).\end{aligned}$$

Reconsidering the derivatives of the density function, equation (A7) is equivalent to

$$\text{(A12)}\quad \begin{aligned}\triangle l_{2,1}=&\frac{1}{6\left(d\mu_1/dx\right)^2}\left[\eta_3\left(x^2-1-\frac{d^3\mu_1/dx^3}{d\mu_1/dx}+\frac{3\,x\left(d^2\mu_1/dx^2\right)}{d\mu_1/dx}+\frac{3\left(d^2\mu_1/dx^2\right)^2}{\left(d\mu_1/dx\right)^2}\right)\right.\\ &\left.+\frac{d\eta_3}{dx}\left(-2x-\frac{3\left(d^2\mu_1/dx^2\right)}{d\mu_1/dx}\right)+\frac{d^2\eta_3}{dx^2}\right].\end{aligned}$$

Similarly, the second part of (A6) will be calculated

$$\text{(A13)}\qquad \triangle l_{2,2} = \frac{1}{8\,n_x}\frac{d}{dx}\left(\frac{n_x}{d\mu_1/dx}\underbrace{\left[\frac{1}{n_x}\frac{d}{dx}\left(\frac{\eta_2\,n_x}{d\mu_1/dx}\right)\right]}_{*}^{2}\right).$$

For (*) we can use the derivation of the first order adjustment in *Wilde* (2001), so we get

$$\text{(A14)}\qquad \begin{aligned} \triangle l_{2,2} &= \frac{1}{8\,n_x}\frac{d}{dx}\left(\frac{n_x}{d\mu_1/dx}\left[\frac{-x\cdot\eta_2}{d\mu_1/dx} + \frac{d\eta_2/dx}{d\mu_1/dx} - \frac{\eta_2\cdot d^2\mu_1/dx^2}{\left(d\mu_1/dx\right)^2}\right]^2\right) \\ &= \frac{1}{8}\left[\underbrace{\frac{1}{n_x}\frac{d}{dx}\left(\frac{n_x}{\left(d\mu_1/dx\right)^3}\right)}_{\mathrm{I}}\cdot\left(-x\cdot\eta_2 + \frac{d\eta_2}{dx} - \frac{\eta_2\cdot d^2\mu_1/dx^2}{d\mu_1/dx}\right)^2\right. \\ &\quad \left. + \frac{1}{\left(d\mu_1/dx\right)^3}\underbrace{\frac{d}{dx}\left(\left[-x\cdot\eta_2 + \frac{d\eta_2}{dx} - \frac{\eta_2\cdot d^2\mu_1/dx^2}{d\mu_1/dx}\right]^2\right)}_{\mathrm{II}}\right]. \end{aligned}$$

For term (I) we obtain

$$\text{(A15)}\qquad \frac{1}{n_x}\frac{d}{dx}\left(\frac{n_x}{\left(d\mu_1/dx\right)^3}\right) = \frac{-x}{\left(d\mu_1/dx\right)^3} - 3\,\frac{\left(d^2\mu_1/dx^2\right)}{\left(d\mu_1/dx\right)^4}$$

Calculating (II) leads to

$$\text{(A16)}\qquad \begin{aligned} &\frac{d}{dx}\left(\left[-x\cdot\eta_2 + \frac{d\eta_2}{dx} - \frac{\eta_2\cdot d^2\mu_1/dx^2}{d\mu_1/dx}\right]^2\right) = 2\left(-x\cdot\eta_2 + \frac{d\eta_2}{dx} - \frac{\eta_2\cdot d^2\mu_1/dx^2}{d\mu_1/dx}\right) \\ &\quad\cdot\left(-\eta_2 - x\frac{d\eta_2}{dx} + \frac{d^2\eta_2}{dx^2} - \frac{d\eta_2}{dx}\frac{d^2\mu_1/dx^2}{d\mu_1/dx} - \eta_2\frac{d^3\mu_1/dx^3}{d\mu_1/dx} + \eta_2\,\frac{d^2\mu_1}{dx^2}\frac{d^2\mu_1/dx^2}{\left(d\mu_1/dx\right)^2}\right). \end{aligned}$$

Therewith, we get for equation (A13)

$$\text{(A17)}\qquad \begin{aligned} \triangle l_{2,2} = {}& \frac{1}{8\left(d\mu_1/dx\right)^3}\left[\left(-x - 3\,\frac{d^2\mu_1/dx^2}{d\mu_1/dx}\right)\left(\eta_2\left[-x - \frac{d^2\mu_1/dx^2}{d\mu_1/dx}\right] + \frac{d\eta_2}{dx}\right)^2\right. \\ & + 2\left(\eta_2\left[x + \frac{d^2\mu_1/dx^2}{d\mu_1/dx}\right] - \frac{d\eta_2}{dx}\right)\left(\eta_2\left[1 + \frac{d^3\mu_1/dx^3}{d\mu_1/dx} - \frac{\left(d^2\mu_1/dx^2\right)^2}{\left(d\mu_1/dx\right)^2}\right]\right. \\ & \left.\left. + \frac{d\eta_2}{dx}\left[x + \frac{d^2\mu_1/dx^2}{d\mu_1/dx}\right] - \frac{d^2\eta_2}{dx^2}\right)\right]. \end{aligned}$$

Thus, our primary equation (A6) can be expressed by the equations (A12) and (A17). Until this point, we only assumed the systematic factor to be normal distributed. For the contained conditional moments we get

$$\text{(A18)} \quad \begin{aligned} &\mu_1 = \sum_{i=1}^{J} w_i \cdot p_i(x)\,,\ \eta_2 = \sum_{i=1}^{J} w_i^2 \big(p_i(x) - p_i^2(x)\big) \quad \text{and} \\ &\eta_3 = \sum_{i=1}^{J} w_i^3 \big[p_i(x) - 3p_i^2(x) + 2p_i^3(x)\big]. \end{aligned}$$

Now, we perform the second order adjustment with respect to the probability of default

$$\text{(A19)} \quad p(x) = N(y), \quad \text{with} \quad y = \frac{N^{-1}(PD) - \sqrt{\varrho} \cdot x}{\sqrt{1-\varrho}} \quad \text{and} \quad s = \frac{\sqrt{\varrho}}{\sqrt{1-\varrho}},$$

of the *Vasicek* model. Having a closer look at (A17) and the conditional moments, we find that the following derivatives are needed

$$\text{(A20)} \quad \frac{d\big(p(x)\big)}{dx} = -s \cdot n(y), \frac{d^2\big(p(x)\big)}{dx^2} = -s^2 \cdot y \cdot n(y), \frac{d^3\big(p(x)\big)}{dx^3} = -s^3 \cdot n(y)\big[y^2 - 1\big],$$

$$\text{(A21)} \quad \frac{d\big(p^2(x)\big)}{dx} = -2\,s \cdot N(y) \cdot n(y), \frac{d^2\big(p^2(x)\big)}{dx^2} = 2\,s^2 \cdot n(y)\big[n(y) - N(y) \cdot y\big],$$

$$\text{(A22)} \quad \frac{d\big(p^3(x)\big)}{dx} = -3\,s \cdot N^2(y) \cdot n(y), \frac{d^2\big(p^3(x)\big)}{dx^2} = 3\,s^2 \cdot N(y) \cdot n(y)\big[2\,n(y) - N(y) \cdot y\big].$$

Finally, we just have to use equations (A12), (A17)–(A22) in order to perform (A6). To simplify the illustration, we will reproduce the complete formula only for a homogeneous portfolio

$$\text{(A23)} \quad \begin{aligned} \triangle l_2 = {} & \frac{1}{6J^2 s^2 n_y^2}\Big[\big(x^2 - 1 + s^2 + 3x\,s\,y + 2s^2y^2\big)\Big(N_y - 3N_y^2 + 2N_y^3\Big) \\ & + s\,n_y\big(2x + 3s\,y\big)\Big(1 - 6N_y + 6N_y^2\Big) - s^2 n_y\big(y - 6\big[N_y y - n_y\big] + 6N_y\big[N_y y - 2n_y\big]\big)\Big] \\ & - \frac{1}{8J^2 s^3 n_y^3}\Big[\big(-x - 3s\,y\big)\Big(\Big[N_y - N_y^2\Big]\big[-x - s\,y\big] - s\,n_y\big[1 - 2N_y\big]\Big)^2 \\ & + 2\Big(\Big[N_y - N_y^2\Big]\big[x + s\,y\big] + s\,n_y\big[1 - 2N_y\big]\Big) \\ & \cdot \Big(\Big[N_y - N_y^2\Big]\big[1 - s^2\big] - s\,n_y\big[1 - 2N_y\big]\big[x + s\,y\big] + s^2 n_y\big[y + 2\big(n_y - N_y y\big)\big]\Big)\Big], \end{aligned}$$

with $N_y = N(y)$, $n_y = n(y)$, $y = \dfrac{N^{-1}(PD) - \sqrt{\varrho} \cdot x}{\sqrt{1-\varrho}}$, $s = \dfrac{\sqrt{\varrho}}{\sqrt{1-\varrho}}$, $x = q_{1-z}(\tilde{x}_T)$.

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Summary

Concentration Risk under Pillar 2: When are Credit Portfolios Infinitely Fine Grained?

The ongoing debate concerning credit concentration risk is mainly driven by the requirements on credit risk management due to Pillar 2 of Basel II since risks (e.g. concentration risk) that are not fully captured by Pillar 1 should be adequately considered in the banks' risk management. This instruction is indeed relevant since quantifying credit portfolio risk in Pillar 1 is based on an Asymptotic Single Risk Factor (ASRF) framework in which concentration risk is not covered. Against the background of the ASRF model, we determine the number of credits up to which concentration risk leads to a significant estimation error so that the assumption of an infinitely fine grained portfolio is inadequate. We conclude that the critical portfolio size varies from 22 up to 35,986 debtors, dependent on assets correlation and probability of default. Using a modified valuation function (granularity adjustment) it is possible to reduce the critical number of credits by averaged 83.04%. (JEL G21, G28)

Zusammenfassung

Konzentrationsrisiken unter Basel II: Wann sind Kreditportfolios unendlich granular?

Die Diskussion hinsichtlich Kredit-Konzentrationsrisiken wird hauptsächlich durch die Anforderungen an das Kreditrisikomanagement getrieben, wie diese in Säule 2 von Basel II formuliert sind. So fordert Säule 2, dass Risiken (z.B. Konzentrationsrisiken), die im Rahmen des Risikomanagements einer Bank nicht vollständig durch Säule 1 erfasst werden, adäquat zu berücksichtigen sind. Diese Forderung erscheint für Konzentrationsrisiken plausibel, da die Ermittlung des Kreditportfoliorisikos gemäß Säule 1 auf einem Asymptotischen Ein-Faktor-(engl.: Asymptotic Single Risk Factor, ASRF)Modell basiert, mit dem solche Risiken nicht abgedeckt werden. Vor dem Hintergrund des ASRF-Modells wird im vorliegenden Beitrag die kritische Größe von Kreditportfolios ermittelt, bis zu der Konzentrationsrisiken zu einem signifikanten Schätzfehler führen, sodass die Annahme unendlicher Granularität für entsprechend kleine Portfolios nicht adäquat ist. Unsere Untersuchung zeigt, dass die kritische Portfoliogröße – abhängig von der Kor-

relation zwischen den Kreditnehmerrenditen und der Kreditnehmerausfallwahrscheinlichkeit – von 22 bis 35.986 Krediten variiert. Durch Modifikation der Berechnungsfunktion (Granularitätsanpassung) kann die kritische Kreditanzahl um durchschnittlich 83,04 % gesenkt werden.