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Which is the Correct Discount Rate? Arithmetic Versus Geometric Mean

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Abstract

The paper revisits the two major concepts for average historical returns, i. e., the arithmetic mean and the geometric mean, in order to clarify which approach must be used for which application. Conducting a rigorous derivation with a geometric Brownian motion, we can explain that the appropriate discount rate refers to the mean discrete return and, therefore, to the arithmetic mean rather than the often wrongly applied geometric mean. Likewise, the prominent CAPM relationship between the expected asset return and the expected market return is only valid for the arithmetic mean rather than the geometric mean. Using historical data for the German stock index, we illustrate that an inconsistent application can cause severe deviations from the meaningful ex-ante expected performance of an asset, the true discount rate, the true CAPM risk-adjusted return, and the intended performance scenarios of packaged retail and insurance-based investment products (PRIIPs) within the key information documents (KIDs).

Keywords: Asset returns, Arithmetic mean, Geometric mean, Geometric Brownian Motion, Discount rate, CAPM, PRIIPs

JEL Classification: G12

I. Introduction

When estimating the average return from historical data, there are two established suggestions: the arithmetic mean and the geometric mean. In the context of many practical applications, either the first or the second alternative for the average return is confidently taken without critically challenging whether this choice is appropriate or not. This is surprising because the numerical magnitude between the arithmetic and geometric mean can sharply differ; e. g., for annual equity returns, there are deviations of about 200 bps (basis points) easily observ-

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able. In particular, the average return especially matters when dealing with the following two applications:

First, the estimation of expected returns to indicate a reasonable asset performance for a future investment. Clearly, a six percent expected return determined by the arithmetic mean might suggest something else than a four percent return resulting from the geometric mean. For example, the German Equity Institute (DAI) frequently publishes the DAX triangle in which historical (annual) returns for arbitrarily long investment periods ranging from one year to fifty years are reported. By default, the geometric mean is supposed as a measure for the average return.

Second, the discounted cash flow (DCF) method requires knowledge of the consistent discount rate. Again, it is a major difference using the geometric mean rather than the arithmetic mean for discounting purposes. In this context, risk-adjusted discount rates are typically obtained from the well-known relationship of the capital asset pricing model (CAPM). Hence, the additional question arises whether the relationship between the expected asset return and the expected market return refers to the arithmetic mean or the geometric mean. In practice, we can find advocates for both views. Proponents of the geometric mean include among others *Damodaran* (2013) and *Koller* et al. (2015), whereas *Brealey* et al. (2011) suggest the use of the arithmetic mean, respectively.

The aim of this paper is to clarify for which application which return concept should be used. As a methodic foundation, we refer to asset price returns that are governed by a geometric Brownian motion. This continuous-time process considers a normally distributed return over an instance of time and is known from option pricing theory (see *Black/Scholes* (1973)). Carrying out a formal derivation, we can relate the various return concepts to the corresponding applications.

In particular, we obtain the following findings: (i) The expected discrete return corresponds to the arithmetic mean, while the expected log return has the character of the geometric mean. (ii) The expected discrete return (arithmetic mean) is the return that indicates the expected wealth of the investment and is, therefore, the right choice to reveal the expected wealth increase in contrast to the expected log return (geometric mean). (iii) The appropriate discount rate within a DCF approach is the expected discrete return (arithmetic mean) rather than the geometric mean. (iv) Conducting a consistent CAPM derivation, we see that the prominent relationship between the expected individual asset return and the expected market return through its beta factor only holds for the expected discrete returns (arithmetic means) but not for the expected log returns (geometric means). (v) Taking a closer look at the packaged retail and insurance-based investment products' (PRIIPs) regulatory technical standards (RTS), we detect a wrong use of expected returns. This is because the expected log return (geometric mean) is applied for performance scenarios with an additional but superfluous variance correction. This finding is in line with *Graf* (2019). For a consistent application of performance scenarios, the expected discrete return (arithmetic mean) could be used in conjunction with the variance correction.

This, at first glance, theoretical outcome has strong practical implications. Supposing the wrong return concept typically results in major differences and pricing errors: Based on illustrative data for the German stock index DAX from August 1988 to December 2019, we obtain a deviation between the arithmetic mean and the geometric mean equal to 2.24 percentage points (pps). When using these returns as discount rates, this difference carries over to a mispricing of more than 30 % for a perpetual (non-growing) expected cash flow stream. When determining expected returns with the CAPM, the application of the formula as a wrong relationship between geometric means results in a substantial deviation of more than 10 pps for assets with a high level of idiosyncratic risk. For PRIIPs, we find for an illustrative participation certificate on the DAX with a recommended holding period (RHP) of 30 years performance values according to the RTS which are half the size of the theory-consistent values.

The remainder of the paper is organized as follows: The next section recapitulates well-known principles of asset returns and introduces the geometric Brownian motion as the underlying process of stock prices. In Section III, we analyze the relationship between the expected discrete and log return and typical return estimators. The correct discount rate given asset prices following the geometric Brownian motion is derived in Section IV. In Section V, we clarify which definition of expected returns refers to the prominent return relationship in the CAPM. Section VI deals with the values of the performance scenarios of PRIIPs within the KIDs. Section VII concludes.

II. Theoretical Background

1. Basic Principles of Asset Returns

We set out the notation first and recapitulate basic and well-known principles of asset returns. Throughout the paper, T denotes the total length of the considered time period in years consisting of N equidistant time steps for the subperiod returns. Accordingly, we have $N + 1$ points in time from time 0 to time T. The length Δt of each time step in years is given by

$$\Delta t = \frac{T}{N},$$

and the corresponding frequency is $\frac{1}{\Delta t}$. We further define n as the counter of the observed time steps ranging from 1 to N. For a non-dividend paying asset with price $S_{n\cdot\Delta t}$ at subperiod n, the discrete subperiod return $r_{n,d}$ from time $(n-1)\cdot\Delta t$ to $n\cdot\Delta t$ is

$$r_{n,d} := \frac{S_{n\cdot\Delta t} - S_{(n-1)\cdot\Delta t}}{S_{(n-1)\cdot\Delta t}} = \frac{S_{n\cdot\Delta t}}{S_{(n-1)\cdot\Delta t}} - 1. \tag{1}$$

Analogously, the continuously compounding return $r_{n,log}$ over time step n, hereafter simply referred to as log return, reads

$$r_{n,log} := log\left(\frac{S_{n\cdot\Delta t}}{S_{(n-1)\cdot\Delta t}}\right) = log\left(S_{n\cdot\Delta t}\right) - log\left(S_{(n-1)\cdot\Delta t}\right). \tag{2}$$

As presented in several fundamental textbooks about time series analysis (see e.g. *Tsay* (2005)), both return definitions offer some convenient properties in specific applications. While we can easily show that log returns are additive over time, discrete returns provide the advantage of being additive within portfolios (see e.g. *Dorfleitner* (2002)).

Since both return definitions are used, we regard the magnitude of the difference between the two definitions. Rearranging Equation (2) and substituting Equation (1), we can derive the relation between the two definitions of returns as

$$r_{n,log} = log\left(1 + \frac{S_{n\cdot\Delta t}}{S_{(n-1)\cdot\Delta t}} - 1\right) = log\left(1 + r_{n,d}\right), \tag{3}$$

where $1 + r_{n,d}$ denotes the discrete gross return of the asset. Considering Equation (3), we know from the properties of logarithmic functions that log returns are weakly smaller than discrete returns. For a return of zero, both return definitions coincide but the difference rises with a higher distance of the return from zero. To give an impression of the magnitude of differences between the two definitions, we provide numerical examples for Equation (3) in Table 1. For discrete returns ranging from –20 % to 20 %, we present the corresponding log return and the respective difference between both definitions of returns. Table 1 confirms that differences between both return definitions are small for returns close to zero. We find only marginal differences of up to 2 bps for discrete returns between –2 % and 2 %, which is in line with the findings of *Dorfleitner* (2003). We can categorize returns of that size as typical daily returns. Accordingly, when estimating discrete or log returns from daily data, at first glance, no big differences become apparent. However, after linearly annualizing the daily returns by factor 250, for example, the marginal differences might result in larger discrepancies. Table 1 indicates that for greater positive and negative returns, the difference becomes more striking. While for a discrete return of 5 % or –5 %, the difference amounts to 12 or 13 bps, respectively, it even equals 1.77 pps or 2.31 pps for a return of 20 % or –20 %, respectively. We conclude that for typical annual returns or total returns observed over several years, substantial differences among the two definitions exist. We summarize the findings in:

Result 1 (Relationship between discrete and log returns) While differences between discrete and log returns are of marginal size for returns close to zero, they become substantial for returns deviating from zero. Thereby, log returns are always smaller than their corresponding discrete returns.

Table 1

Estimation of Discrete and Log Returns

Discrete Return $r_{n,d}$	**Log Return** $\log(1 \quad {}_{n\,d})$	**Difference**
–20.00 %	–22.31 %	2.31 pps
–15.00 %	–16.25 %	1.25 pps
–10.00 %	–10.53 %	0.54 pps
–5.00 %	–5.12 %	0.13 pps
–2.00 %	–2.02 %	0.02 pps
–1.00 %	–1.00 %	0.01 pps
–0.50 %	–0.50 %	0.00 pps
0.00 %	0.00 %	–
0.50 %	0.49 %	0.00 pps
1.00 %	0.99 %	0.00 pps
2.00 %	1.98 %	0.02 pps
5.00 %	4.87 %	0.12 pps
10.00 %	9.53 %	0.47 pps
15.00 %	13.97 %	1.02 pps
20.00 %	18.23 %	1.77 pps

Notes: Table 1 presents log returns for given values of discrete returns from –20 % to 20 % according to Equation (3). We measure the difference as the discrete minus the corresponding log return.

2. *Geometric Brownian Motion as Stock Price Process*

The goal of this section is the introduction of a well-known and established stochastic process for stocks to relate known expected returns to familiar arithmetic and geometric averages. For this purpose, we consider a geometric Brownian motion in continuous time, which is used for Black-Scholes option pricing, among other applications. The length of the introduced time step Δt from the previous section converges to *dt*, which denotes an infinitesimally small time step (see *Hull* (2017)). The underlying process describes the discrete return over this infinitesimal short time interval. Under the assumption of normally distributed discrete returns, we can write the stochastic differential equation in the usual way as

$$\frac{dS}{S} = \mu_S \cdot dt + \sigma_S \cdot dz, \tag{4}$$

where $\frac{dS}{S}$ has the character of the discrete return of an asset over time period *dt* and $\mu_S \cdot dt$ denotes the expected discrete return per annum (p. a.) linearly scaled for subperiod *dt*. The volatility p. a. of the returns is given by σ_S, and *z* denotes a standard Wiener process. In principle, the expected value $E(r_{n,d})$ of the discrete return over an arbitrary subperiod length $\Delta t > 0$ from time $t - \Delta t$ to t with $t = n \cdot \Delta t$ is

$$E(r_{n,d}) = E\left(\frac{S_t - S_{t-\Delta t}}{S_{t-\Delta t}}\right).$$

As we will show in Result 5 in Section III, the annualized expected return depends on time period length Δt. In the special case that Δt tends to zero, the annualized expected return $\frac{E(r_{n,d})}{\Delta t}$ converges to μ_S. Therefore, we can be more precise and denote μ_S as the expected (annualized, short-term) discrete return.

The log return for this process follows from Ito's Lemma (see e. g. *Hull* (2017)) and can be formulated by the following stochastic differential equation

$$dlog(S) = \left(\mu_S - \frac{1}{2}\sigma_S^2\right) \cdot dt + \sigma_S \cdot dz. \tag{5}$$

In this representation, $dlog(S)$ has the character of the log return over time period *dt*. Solving the stochastic differential equations in Equation (4) or equivalently Equation (5), we obtain the well-known solution for the stock price

$$\tilde{S}_T = S_0 \cdot e^{\left(\mu_S - \frac{1}{2}\sigma_S^2\right) \cdot T + \sigma_S \cdot \tilde{z}_T}. \tag{6}$$

Since the increments of the Wiener process are normally distributed, Equation (6) implies that the stock price $\tilde{S}_T$ is log-normally distributed with a normally distributed Wiener process $\tilde{z}_T$ with zero mean and variance T. The expected value of the log return over a discrete subperiod Δt reads

$$\begin{aligned} E\left(r_{n,log}\right) = E(\Delta log(S)) &= E\left(log\left(S_t\right) - log\left(S_{t-\Delta t}\right)\right) \\ &= E\left(log\left(S_{t-\Delta t} \cdot e^{\left(\mu_S - \frac{1}{2}\sigma_S^2\right)\cdot \Delta t + \sigma_S \cdot \tilde{z}_{\Delta t}}\right) - log\left(S_{t-\Delta t}\right)\right) \\ &= E\left(\left(\mu_S - \frac{1}{2}\sigma_S^2\right)\cdot \Delta t + \sigma_S \cdot \tilde{z}_{\Delta t}\right) \\ &= \left(\mu_S - \frac{1}{2}\sigma_S^2\right)\cdot \Delta t. \end{aligned} \tag{7}$$

Equation (7) denotes the expected and linearly adjusted log return for any desired length of time step Δt. We define the expected log return p.a. as μ_{logS} such that

$$\mu_{logS} \cdot \Delta t := \left(\mu_S - \frac{1}{2}\sigma_S^2\right)\cdot \Delta t. \tag{8}$$

From Equation (8), we derive the relationship between the expected (short-term) discrete and expected log return, which we summarize in:

Result 2 (Relationship between expected discrete and log returns) The annualized expected (short-term) discrete return μ_S exceeds the expected annualized log return by μ_{logS} half of the annual variance of returns, so that we can write

$$\mu_S = \mu_{logS} + \frac{1}{2}\sigma_S^2. \tag{9}$$

For a positive volatility, the expected discrete return μ_S must be greater than the expected log return μ_{logS} and the difference increases with the volatility of the underlying asset. To get a sense of the magnitude of the difference, we provide a numerical example which is based on historical closing prices of the DAX performance index. We estimate the right-hand side of Equation (9) from empirical data to determine the estimated annualized expected discrete return $\hat{\mu}_S$ according to

$$\hat{\mu}_S = \hat{\mu}_{logS} + \frac{1}{2}\hat{\sigma}_S^2. \tag{10}$$

We employ the simple arithmetic mean to derive the estimator for μ_{logS} and the well-known formula to derive the sample variance of returns. The formulas read

$$\hat{\mu}_{logS} = \frac{1}{N}\sum_{n=1}^{N} r_{n,log} \cdot \frac{1}{\Delta t} \tag{11}$$

$$\hat{\sigma}_S^2 = \frac{1}{N-1}\sum_{n=1}^{N}\left(r_{n,log} - \hat{\mu}_{logS}\cdot\Delta t\right)^2 \cdot \frac{1}{\Delta t}. \tag{12}$$

First, we analyze whether $\hat{\mu}_{logS}$ constitutes an unbiased estimator for μ_{logS}, given the geometric Brownian motion as the underlying stock price process. We take expectations of Equation (11) and rearrange such that

$$\begin{aligned}
\mathrm{E}(\hat{\mu}_{logS}) &= E\left(\frac{1}{N}\sum_{n=1}^{N}\tilde{r}_{n,log}\cdot\frac{1}{\Delta t}\right)\\
&= \frac{1}{N\cdot\Delta t}\sum_{n=1}^{N}E\left(\tilde{r}_{n,log}\right)\\
&= \frac{1}{T}\sum_{n=1}^{N}E\left(log\left(\frac{S_{n\cdot\Delta t}}{S_{(n-1)\cdot\Delta t}}\right)\right)\\
&= \frac{1}{T}\sum_{n=1}^{N}E\left(log\left(\frac{S_0\cdot e^{\left(\mu_S-\frac{1}{2}\sigma_S^2\right)\cdot n\cdot\Delta t+\sigma_S\cdot\tilde{z}_{n\cdot\Delta t}}}{S_0\cdot e^{\left(\mu_S-\frac{1}{2}\sigma_S^2\right)\cdot(n-1)\cdot\Delta t+\sigma_S\cdot\tilde{z}_{(n-1)\cdot\Delta t}}}\right)\right)\\
&= \frac{1}{T}\sum_{n=1}^{N}E\left(\left(\mu_S-\frac{1}{2}\sigma_S^2\right)\cdot\Delta t+\sigma_S\cdot\left(\tilde{z}_{n\cdot\Delta t}-\tilde{z}_{(n-1)\cdot\Delta t}\right)\right)\\
&= \mu_S-\frac{1}{2}\sigma_S^2 = \mu_{logS},
\end{aligned}$$

which we summarize in:

Result 3 (Properties of the arithmetic mean of log returns) The estimator $\hat{\mu}_{logS}$, which is the annualized arithmetic mean of log returns, is an unbiased estimator of the expected annualized log return μ_{logS}. The estimator is independent of the considered subperiod length Δt of the underlying data. Observing a sample of prices from S_0 to $S_T = S_{N\cdot\Delta t}$, $\hat{\mu}_{logS}$ is fixed and not path-depending.

Turning to the numerical example, we estimate the first two moments of the time series of log returns, which is based on daily prices of the DAX performance index from August 1988 to December 2019 under the assumption of 240 trading days per year. The time series is obtained from Thomson Reuters Datastream. Table 2 displays the results. $\hat{\mu}_{logS}$ is equal to 7.13 % and $\hat{\sigma}_S^2$ takes on a value of 4.47 %, which corresponds to a variance correction $\frac{1}{2}\hat{\sigma}_S^2$ of 2.24 %. Using Equation (10), we find that $\hat{\mu}_S$ is equal to 9.37 %. Thus, even for assets or indexes with usual volatility, such as $\hat{\sigma}_S = 21.15\,\%$, substantial differences between both expected return definitions exist. As a rule of thumb, we can state that a volatility of 20 % p. a. implies a difference between the expected annualized discrete return $\hat{}$ and log return $\hat{\mu}_{logS}$ of 2 pps. Since the variance correction increases with the volatility of the underlying and considering our conservative example, the difference is even higher for many individual stocks and when expected returns are expressed as total returns over multiple years.

III. Arithmetic and Geometric Mean of Discrete Returns

With the geometric Brownian motion set up as the underlying stock price process in the previous section, we can further address the issue of how both the expected discrete and expected log return, μ_S and μ_{logS}, respectively, fit to typical return estimators such as the arithmetic and geometric mean of discrete returns.

1. Geometric Mean of Discrete Returns

To begin with, we analyze properties of the geometric mean of discrete returns and relate it to the estimators of the expected returns μ_S and μ_{logS} from the considered stock price process (geometric Brownian motion) in Section II. With regards to the notation introduced in the previous section, we can write the annualized geometric mean $\overline{r}_{geo}$ with frequency $\frac{1}{\Delta t}$ according to

$$\overline{r}_{geo} = \sqrt[T]{\prod_{n=1}^{N}(1+r_{n,d})} - 1. \tag{13}$$

The idea behind $\overline{r}_{geo}$ is a constant (discretely compounding) return for each subperiod for given initial price S_0 and final stock price $S_{N\cdot\Delta t}$

$$S_0 \cdot \left(1+\overline{r}_{geo}\right)^{N\cdot\Delta t} = S_{N\cdot\Delta t}\,.$$

As demonstrated by several authors before (see e.g. *May* (2019)), we can depict the relationship between the geometric mean $\overline{r}_{geo}$ of discrete returns and the estimator for the arithmetic mean $\hat{\mu}_{logS}$ of log returns. For this purpose, we consider the log representation $\overline{r}_{log-geo}$ of the geometric mean $\overline{r}_{geo}$, which reads

$$\overline{r}_{log-geo} := log(1+\overline{r}_{geo}).$$

Substituting $\overline{r}_{geo}$ by the expression in Equation (13), we can show that

$$\begin{aligned}\overline{r}_{log-geo} &= log\left(1+\sqrt[T]{\prod_{n=1}^{N}(1+r_{n,d}\,)}-1\right)\\ &= \frac{1}{T}log\left(\prod_{n=1}^{N}(1+r_{n,d}\,)\right)\\ &= \frac{1}{T}\sum_{n=1}^{N}log\,(1+r_{n,d}\,)\\ &= \frac{1}{T}\sum_{n=1}^{N}r_{n,log} \;=\hat{\mu}_{logS}\,.\end{aligned}$$

We summarize the result in:

Result 4 (Properties of the geometric mean of discrete returns) The log representation $\overline{r}_{log-geo}$ of the geometric mean of discrete returns equals the linearly annualized arithmetic mean $\hat{\mu}_{logS}$ of log returns. According to Result 1, the geometric mean $\overline{r}_{geo}$ of discrete returns is closely related to $\overline{r}_{log-geo}$ and converges from above against μ_{logS} for returns approaching zero. Therefore, we conclude

$$\overline{r}_{geo} \gtrsim \overline{r}_{log-geo} \rightarrow \mu_{logS}\,.$$

2. *Arithmetic Mean of Discrete Returns*

Turning to the arithmetic mean of discrete returns, we derive the annual mean as the average of discrete, linearly annualized subperiod returns $r_{n,d}\cdot\frac{1}{\Delta t}$ according to

(14) $$\overline{r}_a \;= \frac{1}{N}\sum_{n=1}^{N}r_{n,d}\cdot\frac{1}{\Delta t} = \frac{1}{T}\sum_{n=1}^{N}r_{n,d}\,.$$

As before, we analyze the unbiasedness of the estimator, i.e. whether $\overline{r}_a$ constitutes an unbiased estimator of μ_S for a stock governed by a geometric Brownian motion introduced in Section II. We take expectations of Equation (14) first and substitute the definition of discrete returns from Equation (1), which gives

$$\begin{aligned}\mathrm{E}(\overline{r}_a\,) &= \frac{1}{N}\sum_{n=1}^{N}E(\tilde{r}_{n,d}\,)\cdot\frac{1}{\Delta t}\\ &= \frac{1}{N}\sum_{n=1}^{N}E\left(\frac{\tilde{S}_{n\cdot\Delta t}}{\tilde{S}_{(n-1)\cdot\Delta t}}-1\right)\cdot\frac{1}{\Delta t}.\end{aligned}$$

We insert the expression for the asset price from Equation (6) for corresponding points in time such that

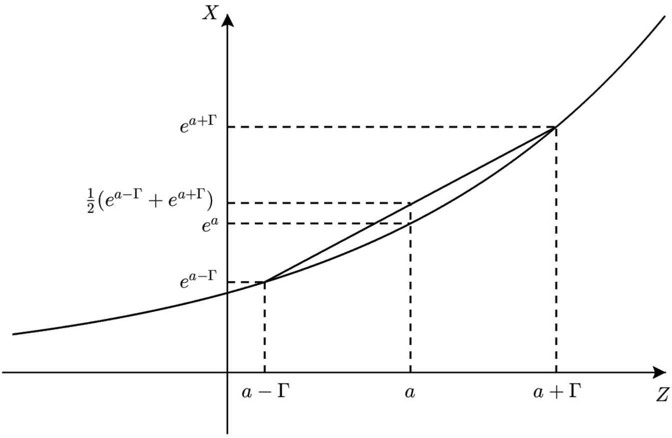
$$
\begin{aligned}
\mathrm{E}(\overline{r}_a) &= \frac{1}{N}\sum_{n=1}^{N} E\left(\frac{S_0 \cdot e^{\left(\mu_S - \frac{1}{2}\sigma_S^2\right)\cdot n\cdot\Delta t + \sigma_S \cdot \tilde{z}_{n\cdot\Delta t}}}{S_0 \cdot e^{\left(\mu_S - \frac{1}{2}\sigma_S^2\right)\cdot (n-1)\cdot\Delta t + \sigma_S \cdot \tilde{z}_{(n-1)\cdot\Delta t}}} - 1\right)\cdot\frac{1}{\Delta t} \\
&= \frac{1}{T}\sum_{n=1}^{N} E\left(e^{\left(\mu_S - \frac{1}{2}\sigma_S^2\right)\cdot\Delta t + \sigma_S\cdot\left(\tilde{z}_{n\cdot\Delta t} - \tilde{z}_{(n-1)\cdot\Delta t}\right)} - 1\right).
\end{aligned}
\tag{15}
$$

Note that $\tilde{z}_{n\cdot\Delta t} - \tilde{z}_{(n-1)\cdot\Delta t}$ denotes an increment of the Wiener process over time step Δt, which is normally distributed with zero mean and variance Δt.

To compute the expectation of Equation (15), we can make use of the fundamental relation that the exponential $X = e^Z$ of a normally distributed variable Z with mean a and standard deviation b is log-normally distributed. The expected value of X is then given by

$$
E(X) = E\left(e^Z\right) = e^{E(Z)+\frac{1}{2}Var(Z)} = e^{a+\frac{1}{2}b^2}.
\tag{16}
$$

Figure 1 illustrates this effect and depicts values of the log-normally distributed variable X depending on the normally distributed variable Z. The convex function characterizes the relationship between the variables. For negative and positive deviations Γ from mean a of variable Z, we find that the mean of $e^{a-\Gamma}$ and $e^{a+\Gamma}$ is located on the connection line $\frac{1}{2}e^{a-\Gamma} + \frac{1}{2}e^{a+\Gamma}$ above e^a. Due to the convexity, the average value of X for any arbitrary pair of symmetric deviations from a has a mean equal to $\frac{1}{2}\left(e^{a-\Gamma} + e^{a+\Gamma}\right) > e^a$. Thus, deriving the expected value of a log-normally distributed variable as in Equation (16) requires to account for the convexity correction. The intuition is that for a negative deviation $-\Gamma$ the loss of the stochastic variable $e^a - e^{a-\Gamma}$ is less than the corresponding gain $e^{a+\Gamma} - e^a$ for a symmetric but favorable outcome of the source Z of uncertainty.



Notes: Figure 1 illustrates values of the log-normally distributed variable X as a function of the normally distributed variable Z according to $X = e^{Z}$.

Figure 1: Illustration of the Convexity Correction

Hence, we can use the property that

$$E\left(e^{\sigma_S \cdot (\tilde{z}_{n\cdot\Delta t} - \tilde{z}_{(n-1)\cdot\Delta t})}\right) = e^{\frac{1}{2}\sigma_S^2 \cdot \Delta t}$$

and write Equation (15) as

$$\begin{aligned} \mathrm{E}(\overline{r}_a) &= \frac{1}{T}\left(\sum_{n=1}^{N} e^{\left(\mu_S - \frac{1}{2}\sigma_S^2\right)\cdot\Delta t + \frac{1}{2}\sigma_S^2\cdot\Delta t} - 1\right) \\ &= \frac{1}{T}\left(\sum_{n=1}^{N} e^{\mu_S\cdot\Delta t} - 1\right) \\ &= \frac{N}{T}\cdot\left(e^{\mu_S\cdot\Delta t} - 1\right) \\ &= \frac{e^{\mu_S\cdot\Delta t} - 1}{\Delta t} \underset{\Delta t\to 0}{\to} \mu_S . \end{aligned}$$

We can conclude that the estimator $\overline{r}_a$ depends on the length Δt of the subperiod. For $\Delta t > 0$, $E(\overline{r}_a)$ is not equal to μ_S. We can show by the rule of l'Hospital that $\overline{r}_a$ is an unbiased estimator of μ_S for an infinitesimally small time interval Δt. This is in contrast to the arithmetic mean of log returns, for which $\hat{\mu}_{logS}$ constitutes an unbiased estimator for μ_{logS} regardless of the length Δt. We summarize the finding in:

Result 5 (Properties of the arithmetic mean of discrete returns) The estimator for the linearly annualized arithmetic mean of discrete returns equals μ_S in the limit as the length of time intervals tends to zero

$$\overline{r}_a \underset{\Delta t \to 0}{\longrightarrow} \mu_S \,.$$

To illustrate the theoretical results, we again consider a numerical example based on the price history of the DAX performance index. Table 2 presents estimators derived from different frequencies of the given time series. We consider daily, weekly, monthly, and annually observed closing prices. We assume a total of 240 trading days per year. It follows that one month includes four weeks with five trading days per week. The geometric mean of discrete returns, displayed in the fifth column, is independent of the frequency and takes on a value of 7.40 %. The arithmetic mean of log returns, which coincides with the log representation of the geometric mean, is given in column 6 and equals 7.13 %.

As the theoretical analysis in the previous section revealed, two possible ways to derive the expected discrete return μ_S exist. We compare the values derived from the arithmetic mean $\overline{r}_a$ of discrete returns with $\hat{\mu}_S = \hat{\mu}_{logS} + -\hat{\sigma}$ from Equation (10). While $\hat{\mu}_{logS}$ is independent of its estimation frequency, the estimator of the variance $\hat{\sigma}_S^2$ of log returns depends on the employed frequency $\frac{1}{\Delta t}$. The estimates for $\hat{\mu}_S$ range between 9.36 % and 9.50 % for the frequencies considered.

Alternatively, we can refer to the arithmetic mean $\overline{r}_a$ of discrete returns to derive a value for μ_S. Respective values range from 9.34 % to 9.69 %. The deviations between the values of the two approaches to derive the expected discrete return μ_S are presented in column 4. When the estimations are based on daily prices, we observe a marginal deviation of less than 1 bp. The inaccuracy increases with the length Δt of the time period up to a high deviation of 19 bps for annual subperiods. Therefore, the estimation of μ_S should necessarily be based on higher frequency data, preferably on daily prices. In this case, $\overline{r}_a$ and $\hat{\mu}_S$ coincide for practical purposes.

Table 2

Estimation for Different Frequencies

	$\overline{r}_a$	$\hat{\mu}_S$	Deviation	$\overline{r}_{geo}$	$\hat{\mu}_{logS}$	$\hat{\sigma}_S^2$
Daily prices	9.3694 %	9.3704 %	–0.0010	7.3955 %	7.1348 %	4.4712 %
Weekly prices	9.3910 %	9.3960 %	–0.0049	7.3955 %	7.1348 %	4.5223 %
Monthly prices	9.3386 %	9.3640 %	–0.0254	7.3955 %	7.1348 %	4.2141 %
Annual prices	9.6906 %	9.5007 %	0.1899	7.3955 %	7.1348 %	4.4984 %

Notes: Table 2 presents estimators for varying frequencies based on closing prices of the DAX performance index from August 1988 to December 2019. Daily, weekly, monthly, and annual price frequencies are considered. One year consists of 240 trading days with 48 weeks and five trading days per week. $\overline{r}_a$ denotes the arithmetic mean of discrete returns. The estimator of the expected return $\hat{\mu}_S$ is derived according to Equation (10) as $\hat{\mu}_S = \hat{\mu}_{logS} + \frac{1}{2}\hat{\sigma}_S^2$, where $\hat{\sigma}_S^2$ denotes the variance of log returns p. a. The deviation refers to the difference between $\overline{r}_a$ and $\hat{\mu}_S$. $\overline{r}_{geo}$ denotes the geometric mean of discrete returns, and $\hat{\mu}_{logS}$ is the estimated expected log return, which is derived from the arithmetic mean of log returns according to Equation (11). The estimator is equivalent to the log representation of the geometric mean $\overline{r}_{log-geo}$.

IV. The Proper Discount Rate

In this section, we derive the required rate of return of investors used to discount expected asset prices or expected cash flows appropriately to determine the fair value of assets. Practically speaking, we address the issue of whether the expected discrete return μ_S (proxied by the arithmetic mean $\overline{r}_a$ of discrete returns) or the expected log return μ_{logS} (proxied by the geometric mean $\overline{r}_{geo}$ of discrete returns) is the proper discount rate. According to (one-period) DCF valuation, the present value S_0 must equal the expected cash flows $E\left(\tilde{S}_T\right)$ discounted with the appropriate rate y, which denotes the continuously compounding discount rate

(17) $$S_0 \overset{!}{=} e^{-y \cdot T} \cdot E\left(\tilde{S}_T\right).$$

Hence, we can implicitly solve Equation (17) for y as

$$y = \frac{1}{T} log\left(\frac{E\left(\tilde{S}_T\right)}{S_0}\right).$$

Inserting the geometric Brownian motion from Equation (6) for the asset price S_T, we directly obtain

$$y = \frac{1}{T} log\left(e^{(\mu_{logS} + \frac{1}{2}\sigma_S^2)\cdot T}\right)$$

$$= \mu_{logS} + \frac{1}{2}\sigma_S^2 = \mu_S .$$

We summarize the finding in:

Result 6 (the proper discount rate) The annualized expected discrete return μ_S is the required rate of return and depicts the proper rate to be used for discounting purposes.

Hence, the arithmetic mean $\overline{r}_a$ of discrete returns, which converges to μ_S for short time periods Δt equal to a day, is the appropriate discount rate rather than the geometric mean $\overline{r}_{geo}$, which is close to μ_{logS}.

If the expected log return μ_{logS} is erroneously used as the discount rate without the required variance correction, the resulting asset price S_0 will be overestimated. To see the effects on valuation, we consider the following numerical example. We suppose a perpetual stream of expected cash flows of Euro 100 p.a. The present value PV_0 of the infinite stream of expected cash flows with the continuously compounding discount rate y simplifies to

$$PV_0 = \int_0^T \frac{100}{e^{y\cdot t}} dt = \underset{T\to\infty}{\to} \frac{100}{y}.$$

To illustrate differences in the resulting asset prices, we determine the present values of the streams of cash flows subject to both the annualized expected discrete return μ_S and the log return μ_{logS}. For the magnitude of the respective discount rates, we take the derived parameters from the previous example of the DAX performance index presented in Table 2. With the estimator $\hat{\mu}_S$ of the annualized expected discrete return as discount rate y, we derive the correct asset price S_0 as

$$S_0 = \frac{100}{9.37\%} = 1{,}067.$$

The incorrect asset price S_0 subject to the application of $\hat{\mu}_{logS}$ as the discount rate y is given by

$$S_0 = \frac{100}{7.13\%} = 1{,}402.$$

The difference between the correct and incorrect discount rate of approximately 2 pps results in a valuation difference of Euro 335, or about 31 %. The result indicates that the application of the expected log return μ_{logS}, or the closely related geometric mean $\overline{r}_{geo}$, leads to substantially upwardly biased asset prices in DCF valuation.

Revisiting the return triangle of the DAI, which reports the geometric mean $\overline{r}_{geo}$ over varying investment horizons, we now see that $\overline{r}_{geo}$ is misleading for both discounting purposes and indicating the expected performance.

First, regarding DCF valuation, the given figures would imply too low required returns to justify prices for given expectations. Second, from a performance point of view, the average return an investor obtains is $E(S_T) = S_0 \cdot e^{\mu_S \cdot T}$ and therefore more than the expected log return μ_{logS}, or the geometric mean $\overline{r}_{geo}$, respectively, and needs to be adjusted with the variance correction.

V. Capital Asset Pricing Model

In this section, we formally derive the CAPM relationship with a focus on the particular definition of the expected return. This derivation aims at clarifying the issue whether the expected discrete return μ_S or the expected log return μ_{logS} refers to the prominent return relationship in the CAPM.

In practice, advocates for both views exist. On the one hand, *Damodaran* (2013) claims that "in corporate finance and valuation, at least, the argument for using geometric average premiums as estimates is strong" (p. 367). Also, *Koller* et al. (2015) take the view that the arithmetic mean of discrete returns μ_S results in an upwardly biased market risk premium and conclude that the market risk premium is in the range between the arithmetic and geometric average. On the other hand, proponents claiming μ_S to be the correct estimate are among others *Brealey* et al. (2011). They propose that "if the cost of capital is estimated from historical returns or risk premiums, use arithmetic averages, not compound annual rates of return" (p. 159).

Similarly, we observe that a proxy for the expected market return smaller than the arithmetic mean of discrete returns is frequently used in practice to derive the market risk premium. In line *Stehle* (2004), the Technical Committee for Business Valuation and Economics (FAUB) of the Institut der Wirtschaftsprüfer (IDW) proposes a fixed discount on the empirically estimated arithmetic mean μ_S. The IDW regularly publishes a range of feasible values for the market risk premium derived accordingly.[1] A survey conducted by KPMG (2019) reports that the market risk premium used in practice by German companies lies within the provided bandwidth of the IDW and thus between the estimates of μ_S and μ_{logS}.

[1] https://www.idw.de/idw/idw-aktuell/neue-kapitalkostenempfehlungen-des-faub/120158.

1. Proof of the CAPM

To provide a proof for the CAPM, we adopt a consumption-based framework similar to *Huang/Litzenberger* (1988) and *Cochrane* (2009). We regard a one-period pure exchange economy with points in time $t = 0$ and $t = 1$. There exist units of a single perishable consumption good at both points in time. For a given aggregate endowment, individuals choose their optimal consumption at $t = 0$ and state-contingent claims on consumption for time $t = 1$. The economy is characterized by uncertainty about the state at time $t = 1$. Individuals have a homogeneous probability assessment of the occurrence of state ω of all possible states Ω at $t = 1$, which we denote π_ω. Further, we adopt a time-additive and state-independent utility function, which is increasing in consumption and strictly concave. Satisfying the criteria, we assume a logarithmic utility function. In the competitive economy, we can consider one representative agent for pricing purposes whose endowment equals aggregate individuals' endowments. Hence, lifetime utility is

$$\begin{aligned} u(C_0, C_\omega) &= u_0(C_0) + \delta \cdot u_1(C_\omega) \\ &= log(C_0) + \delta \cdot log(C_\omega), \end{aligned} \tag{18}$$

where C_0 denotes aggregate consumption at time $t = 0$, which is exogenously given and serves as the numeraire throughout. C_ω indicates aggregate consumption in state ω. Following *Cochrane* (2009), we allow for a discount factor δ, which captures the impatience of investors. As a well-known result, the price ϕ_ω at $t = 0$ of a contingent claim for one unit of consumption in state ω is given by

$$\phi_\omega = \delta \cdot \frac{\pi_\omega u_1'(C_\omega)}{u_0'(C_0)}, \quad for\ \omega \in \Omega. \tag{19}$$

Thus, the price S_0 of an asset at $t = 0$ paying $e^{x_{S_\omega}}$ in state ω follows from a weighted average of cash flows $e^{x_{S_\omega}}$ with corresponding state prices ϕ_ω as

$$S_0 = \sum_{\omega \in \Omega} \phi_\omega e^{x_{S_\omega}}. \tag{20}$$

Inserting the expression for ϕ_ω from Equation (19) in Equation (20) results in the well-known basic asset pricing equation

$$S_0 = E\left(\delta \cdot \frac{u_1'(\tilde{C})}{u_0'(C_0)} e^{\tilde{x}_S}\right), \tag{21}$$

in which $\tilde{C}$ and $\tilde{S} = e^{\tilde{x}_S}$ denote aggregate consumption and the payoff of an asset at $t = 1$, respectively.

For the considered economy, the aggregate consumption $\tilde{C}$ at $t = 1$ equals aggregate wealth $\tilde{M}$. The random payoff at time $t = 1$ of the market portfolio is denoted $e^{\tilde{x}_M}$ such that we can summarize:

$$\tilde{C} = \tilde{M} = e^{\tilde{x}_M} . \tag{22}$$

The payoff of the market portfolio $\tilde{M}$, as well as the payoff $\tilde{S}$ of the asset, are log-normally distributed. $\tilde{x}_M = log(\tilde{M})$ and $\tilde{x}_S = log(\tilde{S})$ are bivariate normally distributed with mean $\mathrm{E}(\tilde{x}_M) = \theta_M$ and $\mathrm{E}(\tilde{x}_S) = \theta_S$, standard deviation σ_M and σ_S, respectively, and correlation ρ. Note that the pricing Equation (21) holds for both continuous and discrete states of the state variables $\tilde{x}_S$ and $\tilde{x}_M$.

Given the described economy, we derive an expression for the risk-free interest rate. For this purpose, we consider a risk-free zero bond paying one unit of the consumption good at time $t = 1$ in all possible states of the world such that $e^{x_f} = 1$. Using the asset pricing equation from Equation (21) and the described equivalence of consumption at time $t = 1$ and the payoff of the market portfolio from Equation (22), we derive the price B of the zero bond as

$$B = e^{-r_f} = \delta \cdot E\left(\frac{u_1'\left(e^{\tilde{x}_M}\right)}{u_0'(C_0)} \cdot 1\right),$$

where r_f denotes the continuously compounding risk-free interest rate. Substituting the first derivative of the given utility function results in

$$\begin{aligned} e^{-r_f} &= \delta \cdot E\left(\frac{1/e^{\tilde{x}_M}}{1/C_0}\right) \\ &= \delta \cdot C_0 \cdot E\left(e^{-\tilde{x}_M}\right). \end{aligned}$$

In accordance with Equation (16), it follows from the expected value of the log-normally distributed payoff of the market portfolio that

$$e^{-r_f} = \delta \cdot C_0 \cdot e^{-\theta_M + \frac{1}{2}\sigma_M^2} .$$

Rearranging, we can write the risk-free interest rate as

$$r_f = -log(\delta) - log(C_0) + \theta_M - \frac{1}{2}\sigma_M^2 . \tag{23}$$

With the expression for the risk-free interest rate at hand, we can derive the expected excess return of the market portfolio. According to Equation (21), the price of the market portfolio M_0 at time $t = 0$ is

$$M_0 = \delta \cdot E\left(\frac{u_1'(\tilde{C})}{u_0'(C_0)} \cdot \tilde{C}\right) = \delta \cdot E\left(\frac{u_1'(e^{\tilde{x}_M})}{u_0'(C_0)} \cdot e^{\tilde{x}_M}\right) = \delta \cdot C_0 . \tag{24}$$

We can interpret the time $t = 0$ price $\delta \cdot C_0$ of the market portfolio as the consumption at time $t = 0$ that needs to be given up in order to be able to consume all available consumption goods at time $t = 1$.

We exploit Equation (24) to derive an expression for the log return of the market portfolio, which we denote $\tilde{r}_M^{log}$. The return from time $t = 0$ to time $t = 1$ reads

$$\tilde{r}_M^{log} = log\left(\frac{\tilde{M}}{M_0}\right) = log\left(\frac{e^{\tilde{x}_M}}{\delta \cdot C_0}\right) = \tilde{x}_M - log(C_0) - log(\delta). \tag{25}$$

The expected excess return of the market portfolio $E\left(\tilde{r}_M^{log}\right) - r_f$ implied by Equation (25) and Equation (23) can be written as

$$E\left(\tilde{r}_M^{log}\right) - r_f = E(\tilde{x}_M) - log(C_0) - log(\delta) - \left(-log(\delta) - log(C_0) + \theta_M - \frac{1}{2}\sigma_M^2\right),$$

which simplifies to

$$E\left(\tilde{r}_M^{log}\right) - r_f = \frac{1}{2}\sigma_M^2 . \tag{26}$$

In the next step, we derive an expression for the expected excess return of an asset. Similar to the market portfolio, we first write the price of the asset at time $t = 0$ using the asset pricing equation from Equation (21) as

$$\begin{aligned} S_0 &= \delta \cdot E\left(\frac{u_1'(e^{\tilde{x}_M})}{u_0'(C_0)} \cdot e^{\tilde{x}_S}\right) \\ &= \delta \cdot C_0 \cdot E\left(e^{\tilde{x}_S - \tilde{x}_M}\right). \end{aligned}$$

As the exponential term on the right-hand side is log normal, we obtain[2]

[2] In line with Equation (16), the expected value of $X = e^{Z_1 - Z_2}$ with normally distributed variables Z_1 and Z_2 with mean a_1 and a_2, standard deviation b_1 and b_2, respectively, and correlation ρ is

$$\begin{aligned} E\left(e^{Z_1 - Z_2}\right) &= e^{E\left(e^{Z_1 - Z_2}\right) + \frac{1}{2}Var(Z_1 - Z_2)} \\ &= e^{E(Z_1) + E(-Z_2) + \frac{1}{2}(Var(Z_1) + Var(-Z_2) + 2 \cdot Cov(Z_1, -Z_2))} \\ &= e^{a_1 - a_2 + \frac{1}{2}b_1^2 + \frac{1}{2}b_2^2 - b_1 \cdot b_2 \cdot \rho} . \end{aligned}$$

$$S_0 = \delta \cdot C_0 \cdot e^{\theta_S - \theta_M + \frac{1}{2}\sigma_S^2 + \frac{1}{2}\sigma_M^2 - \sigma_S \cdot \sigma_M \cdot \rho}.$$

We denote the log return of an asset from time $t = 0$ to time $t = 1$ as

$$\begin{aligned}\tilde{r}_S^{log} &= log\left(\frac{e^{\tilde{x}_S}}{S_0}\right) = log\left(\frac{e^{\tilde{x}_S}}{\delta \cdot C_0 \cdot e^{\theta_S - \theta_M + \frac{1}{2}\sigma_S^2 + \frac{1}{2}\sigma_M^2 - \sigma_S \cdot \sigma_M \cdot \rho}}\right) \\ &= -log(\delta) - log(C_0) + \tilde{x}_S - \theta_S + \theta_M - \frac{1}{2}\sigma_S^2 - \frac{1}{2}\sigma_M^2 + \sigma_S \cdot \sigma_M \cdot \rho,\end{aligned}$$

and in expectation

$$E\left(\tilde{r}_S^{log}\right) = -log(\delta) - log(C_0) + \theta_M - \frac{1}{2}\sigma_S^2 - \frac{1}{2}\sigma_M^2 + \sigma_S \cdot \sigma_M \cdot \rho. \tag{27}$$

Using Equation (27) and the expression for the risk-free rate from Equation (23), we can write the following condition

$$E\left(\tilde{r}_S^{log}\right) - r_f + \frac{1}{2}\sigma_S^2 = \sigma_S \cdot \sigma_M \cdot \rho. \tag{28}$$

The CAPM result, i. e., the relation between the expected excess return of an asset and that of the market portfolio can now be formulated with the equations deduced above.

Using Equation (26) and Equation (28), we get

$$\begin{aligned}E\left(\tilde{r}_S^{log}\right) + \frac{1}{2}\sigma_S^2 - r_f &= \frac{\sigma_S \cdot \sigma_M \cdot \rho}{\sigma_M^2} \cdot \left(E\left(\tilde{r}_M^{log}\right) + \frac{1}{2}\sigma_M^2 - r_f\right) \\ &= \beta \cdot \left(E\left(\tilde{r}_M^{log}\right) + \frac{1}{2}\sigma_M^2 - r_f\right),\end{aligned}$$

with the well-known definition for $\beta = \frac{\sigma_S \cdot \sigma_M \cdot \rho}{\sigma_M^2}$. Introducing the expected log return μ_{logS} of an asset and μ_{logM} of the market portfolio, respectively, we find the following fundamental relationship

$$\mu_{logS} + \frac{1}{2}\sigma_S^2 = r_f + \beta \cdot \left(\mu_{logM} + \frac{1}{2}\sigma_M^2 - r_f\right). \tag{29}$$

Using Equation (9), we can alternatively formulate:

Result 7 (CAPM) The CAPM describes the relation between the expected discrete return μ_S of an asset and the expected discrete return μ_M of the market

$$\mu_S = r_f + \beta \cdot \left(\mu_M - r_f\right).$$

When expected log returns μ_{logS} or μ_{logM}, i.e. the arithmetic mean of log returns, or the closely related geometric mean (see Result 2), are used, it is necessary to add the variance corrections to adjust expected returns according to Equation (29).

2. *Analysis of a Potential Inaccuracy*

We now regard a typical inaccuracy that occurs when the CAPM is applied in practice. This is the case when the CAPM is erroneously treated as a relation between expected log returns, μ_{logS} and μ_{logM}, rather than the correct relation between expected discrete returns, μ_S and μ_M. We denote the incorrect expected log return μ_{logS}, and write the inaccurate relationship as

(30) $$\mu'_{logS} = r_f + \beta \cdot \left(\mu_{logM} - r_f\right),$$

while the correct CAPM relation for the expected log return μ_{logS} follows from Equation (29) and reads

(31) $$\mu_{logS} = r_f + \beta \cdot \left(\mu_{logM} + \frac{1}{2}\sigma_M^2 - r_f\right) - \frac{1}{2}\sigma_S^2 .$$

Subtracting Equation (31) from Equation (30), we can quantify the size of the error that is committed as

$$\mu'_{logS} - \mu_{logS} = \frac{1}{2}\left(\sigma_S^2 - \beta \cdot \sigma_M^2\right).$$

In order to analyze the difference, we break down the total variance σ_S^2 of the asset into the systematic variance $\beta^2 \cdot \sigma_M^2$ and the idiosyncratic variance σ_ε^2 according to

(32) $$\sigma_S^2 = \beta^2 \cdot \sigma_M^2 + \sigma_\varepsilon^2 .$$

Equation (32) allows us to write the error as

$$\mu'_{logS} - \mu_{logS} = \frac{1}{2}\left(\sigma_\varepsilon^2 + \beta \cdot (\beta - 1) \cdot \sigma_M^2\right).$$

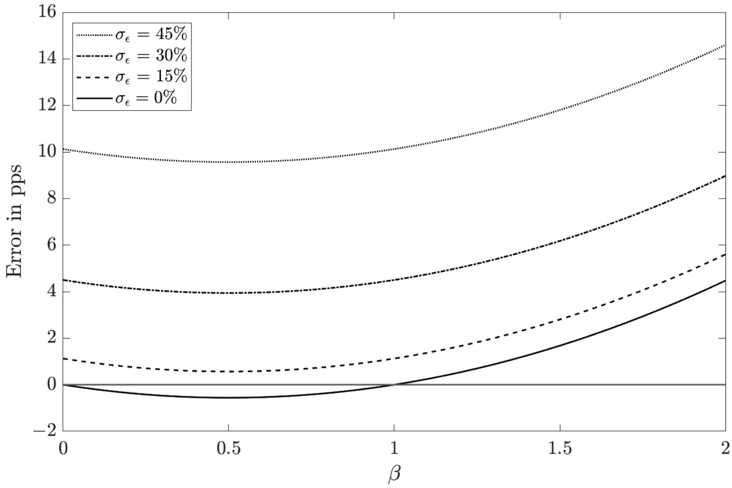
The size of the error increases with the idiosyncratic volatility σ_ε, else equal. The quadratic function $\mu'_{logS} - \mu_{logS}$ in β has its minimum at $\beta = \frac{1}{2}$ with a value of the error of $\frac{1}{2}\left(\sigma_{\varepsilon}^2 - \frac{1}{4}\sigma_M^2\right)$. For $\beta = 0$ and $\beta = 1$ the error equals $\frac{1}{2}\sigma_{\varepsilon}^2$ and further increases with β.

Figure 2 illustrates the size of the error, taking the DAX as the market portfolio with $\hat{\sigma}_M = 21.15\,\%$, as a function of β for three levels of the idiosyncratic risk σ_{ε}. The figure indicates that the error can be negative for small levels of idiosyncratic risk. The maximum negative error for $\sigma_{\varepsilon} = 0\,\%$ and $\beta = \frac{1}{2}$ equals $-\frac{1}{4}\sigma_M^2$ and amounts to approximately –55 bps in our example. The error becomes positive for $\beta > 1$. As the idiosyncratic volatility σ_{ε} of stocks is usually equal or greater than the volatility σ_M of the market, however, we can typically expect a positive error. Hence, the expected log return μ_{logS} of an asset is overestimated when the CAPM is treated as a relation between expected log returns. Motivated by a special but not unrealistic case of a correlation of 0.40 between the returns of the market and an asset and a volatility σ_S of 50 %, we obtain an idiosyncratic volatility of approximately $\sigma_{\varepsilon} = 45\,\%$. We consider this parameter specification as the maximum in Figure 2. Given that high level of idiosyncratic risk, the error can obtain values even greater than 10 pps.

VI. PRIIPs

The next example for a misleading use of expected returns concerns the EU regulation No. 1286/2014 for PRIIPs by the European Commission (2014). According to this regulation, manufacturers have to prepare KIDs for these products. Such PRIIP KIDs characterize the products regarding their market and credit risk, as well as performance scenarios and costs. The RTS of the European Commission (2017) thereby suggest to base calculations on log returns and to apply the arithmetic mean of log returns, i. e. μ_{logS}, to derive the risk summary indicator and performance scenarios. We discuss the correctness of the expected return definition applied with lasting consequences for the derived key figures presented in the KIDs. Using the results from the previous sections, we analyze a methodological inaccuracy regarding the performance calculations of category 2 products following the RTS. Products that belong to category 2 are products whose payoff profile is linear, i. e. a constant multiple, in the performance of the underlying.

According to the RTS, the values for the favorable, moderate, and unfavorable scenario are derived based on respective percentiles of the standard normal distribution adjusted by the approximation by *Cornish/Fisher* (1938) to account for non-normality in the tails of the empirical distribution. As the size of the resulting mistake does not depend on the thereby accounted for skewness and kurtosis of the empirical distribution, we neglect the application of the Cornish Fisher expansion. The values for the performance scenarios V^{RTS} following the RTS rewritten in our notation then read



Notes: Figure 2 illustrates the size of the error that is conducted when the CAPM is taken as a relation between expected log returns as a function of β and for four given levels of idiosyncratic risk σ_ε .The error is expressed in percentage points.

Figure 2: Size of the Error $\mu'_{logS} - \mu_{logS}$ as a Function of β

$$V^{RTS} = e^{\left(\mu_{logS} - \frac{1}{2}\sigma_S^2\right)\cdot T + \sigma_S \cdot x_\alpha \cdot \sqrt{T}}.$$

μ_{logS} corresponds to the annualized arithmetic mean of log returns and T refers to the RHP expressed in years.[3] x_α denotes the α-percentile of the standard normal distribution.[4] Subject to the considered performance scenario, the 10^{th}, 50^{th}, or 90^{th} percentile of the standard normal distribution is applied. We consider the moderate scenario to demonstrate the methodological mistake in the following. With $\alpha = 0.5$ such that $x_\alpha = 0$, the value of the PRIIP in the moderate scenario V_{mod}^{RTS} under the RTS is given by

$$V_{mod}^{RTS} = e^{(\mu_{logS} - \frac{1}{2}\sigma_S^2)\cdot T}. \quad (33)$$

We now derive the value of the PRIIP in the moderate scenario given the stock price process from Section II. In principle, the performance scenarios describe the final wealth $e^{log\left(\frac{\tilde{S}_T}{S_0}\right)}$ at the RHP T subject to the percentile of the re-

[3] The notation of the RTS is adjusted to the notation of this paper and both μ_{logS} and σ_S refer to annualized figures. The computation is based on daily log returns as required by the RTS.

[4] $x_\alpha = \Phi^{-1}(\alpha)$, where $\Phi(\cdot)$ stands for the cumulative distribution function of a standard normally distributed random variable.

spective performance scenario. Using the expression for $\tilde{S}_T$ from Equation (6), we can write the value at the RHP as

$$e^{log\left(\frac{\tilde{S}_T}{S_0}\right)} = e^{\mu_{logS} \cdot T + \sigma_S \cdot \tilde{z}_T} . \tag{34}$$

Since the moderate performance scenario refers to the 50[th] percentile, we derive the value of the PRIIP V_{mod} as the median of Equation (34). Since the median of the Wiener process $\tilde{z}_T$ is zero, we obtain

$$V_{mod} = e^{\mu_{logS} \cdot T} . \tag{35}$$

Using the expected discrete return μ_S to derive the performance value in the moderate scenario instead, we alternatively obtain from Result 2:

$$V_{mod} = e^{(\mu_S - \frac{1}{2}\sigma_S^2) \cdot T} . \tag{36}$$

A comparison of Equation (33) with (36) suggests that the performance values of the RTS may be the result of an erroneous use of the expected log return μ_{logS} rather than the expected discrete return μ_S . Subtracting the variance correction from μ_{logS} is a variance correction twice. We summarize our finding in:

Result 8 (Performance values of PRIIPs) The value of the PRIIP at the RHP according to the RTS in the moderate scenario understates the correct value from Equation (35) due to a variance correction twice. To obtain the theory-consistent value V, the RTS value V^{RTS} needs to be adjusted as follows

$$V = e^{\frac{1}{2}\sigma_S^2 T} \cdot V^{RTS} . \tag{37}$$

This adjustment also holds for the unfavorable and favorable scenarios. Apparently, the values V^{RTS} *of the PRIIP following the RTS are smaller by a factor of* $e^{\frac{1}{2}\sigma_S^2 T}$ *compared to the theory-consistent values V. The valuation error increases both with the RHP T and the volatility of the returns of the underlying.*

To provide a numerical example for the error included in the PRIIPs RTS, we consider a constructed category 2 open-end participation certificate on the DAX performance index. We calculate the values of the performance scenario according to the RTS and the theory-consistent derivation.[5] Further, we assume the empirical distribution of the DAX to be normal and hence neglect the application of the Cornish Fisher expansion for skewness and kurtosis. We regard an RHP of 30 years and consider a usual initial investment of Euro 10,000.

[5] We do not base the calculation of the moments on the past five years of daily returns but use the history of daily closing prices the DAX performance index from 1988 to 2019 such as in the previous examples.

Table 3

Theory-consistent Performance Values and According to the RTS

	RTS		Theory-consistent		
	V^{RTS}	Return	V	Return	Error
Unfavorable	9,856	–0.05 %	19,275	2.21 %	95.55 %
Moderate	43,482	5.02 %	85,032	7.40 %	95.55 %
Favorable	191,828	10.35 %	375,128	12.84 %	95.55 %

Notes: Table 3 presents the performance values of a constructed participation certificate on the DAX performance index calculated according to the RTS and the theory-consistent derivation for an RHP of 30 years. Return denotes the annualized discrete return of the respective performance scenario with an initial investment of Euro 10,000. The error is defined as the percentage by which the theory-compliant performance value V exceeds the corresponding value V^{RTS} of the RTS, which amounts to $e^{\frac{1}{2}\sigma_s^2 T} - 1$.

The performance values, the annualized discrete returns, and the resulting errors, which stand for the percentage by which the real performance value exceeds the corresponding value of the RTS, are given in Table 3. In line with Equation (37), we find that the theory-compliant performance values V exceed the respective values V^{RTS} of the RTS by 95.55 %, i. e. are approximately twice as high. The differences in annualized returns ranging from 2.25 pps in the unfavorable scenario to up to 2.5 pps in the favorable scenario lead to deviations in the values of V and V^{RTS} from Euro 9,500 to Euro 183,300, respectively.

VII. Conclusion

The comparison of the geometric mean of historical returns with the arithmetic mean is closely related to a comparison of the mean log return with the mean discrete return. While both differences seem to be negligible and of minor order for (typical) daily equity returns, the distinction is crucial when those returns are scaled to an annual basis. For a usual equity standard deviation equal to 20 %, both differences between the corresponding two return concepts can easily amount to two percentage points. As a consequence, the correct choice matters when dealing with (i) the appropriate historical performance for the ex-ante expected return of an asset and (ii) the corresponding discount rate when determining the true present value of a future (expected) payoff. As a result of the revisit of established return concepts, we now have a justification for the use of the arithmetic mean of the discrete return (or accordingly the mean discrete return) as the appropriate performance measure for the expected wealth increase of an asset. Likewise, we need to choose these return concepts for the appropriate discount rate rather than the mean log return or the geometric mean. When referring to the CAPM, however, the prominent relationship between expected asset returns and the expected market return is again only valid for the arithmetic mean or the mean of discrete returns. A wrong application of the non-suited average return can lead to severe performance differences and pricing errors. The rigorous comparison of different return concepts conducted in this paper is intended to increase the awareness for the relevance of this issue, to provide a sound explanation for it, and to prevent all kinds of related failures such as the misleading performance scenarios within the PRIIPs key information documents.

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