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Alternative Model Specifications for Implied Volatility Measured by the German VDAX

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I. Introduction

Variance – or equivalently volatility defined as the positive square root of variance – is an important input to option pricing. The classical Black/Scholes/Merton-assumption of stock prices modeled by a geometric Brownian motion with a constant variance is a simplification which has been modified by introducing stochastic return variance. Models of stochastic variance are assumed in the option pricing models by Hull and White (1987), Wiggins (1987) and Heston (1993), among others. Common characteristics in variance behavior which have been observed empirically refer to mean reversion and conditional heteroskedasticity. Since it captures these characteristics and because of its analytical tractability, the mean reverting square root process is a commonly used model of stochastic volatility. In the finance literature, the process was first proposed by Cox, Ingersoll, and Ross (1985b) for describing the dynamic behavior of interest rates.

Apart from option pricing under stochastic variance, models of the variance or volatility dynamics are important for their own sake as well. Modeling and estimating volatility is central in many financial applications. Therefore, it is not surprising that first steps have been taken in the development of a market for volatility (Deutsche Börse AG (1997)). A sound economic reason for trading implied volatility is that it provides a way for market participants to hedge against changes in its level (see e.g. Locarek-Junge and Roth (1998)). As implied volatility is the key input to option prices, the possibility to trade volatility would reduce option portfolio risk which arises from the fact that future implied market vola-

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tility can only be estimated with error. Moreover, trading implied volatility would allow for hedging changes in volatility when replicating options by a position in the underlying and cash. Consequently, preference-free option pricing under stochastic volatility becomes possible (see also Hull and White (1987)). As volatility itself is not an asset providing a payoff stream, contingent claims such as futures and options create useful tradable volatility securities.

Options on volatility have been discussed in Brenner and Galai (1989), Whaley (1993), and Grünbichler and Longstaff (1996) and Brenner, Ou, and Zhang (2000). When theory aims to explain the valuation of contingent claims on volatility, a particular model is needed, representing the underlying volatility process. Whereas Brenner and Galai model volatility conditional on the stock price process focusing on the negative correlation between prices and volatility, Whaley and Grünbichler and Longstaff consider volatility as a single univariate state variable. Whaley's approach relies on geometric Brownian motion as a model of volatility. Grünbichler and Longstaff assume the Cox/Ingersoll/Ross mean reverting square root process. The recent approach by Brenner, Ou, and Zhang prices straddle options where volatility is assumed to follow a mean reverting process.

Mean reversion in volatility is well-documented in the empirical literature. Although various models of stochastic volatility have been proposed, the possibility of discontinuous sample paths has been mostly neglected.[1] In this paper it is argued that, jump diffusion processes which have been proposed and empirically investigated as a description of the dynamic behavior of stock prices, are at least as much appropriate for modeling return volatility. In particular, discontinuities in the volatility series are obviously related to large absolute returns. The investigations by French, Schwert, and Stambaugh (1987) and Schwert (1989), for example, illustrate the possibility of jumps in stock market volatility. This characteristic is also observable for the VIX volatility index as noted by Fleming, Ostdiek, and Whaley (1995).[2] Beinert and Trautmann (1991)

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[1] An exceptional example is the work by *Bookstaber* and *Pomerantz* (1989) which proposes a model where volatility is given as a sum of random jumps. Here we choose a model approach closer related to the mean reverting diffusion model which is the standard volatility model in the literature.

[2] The authors report that there are several "spikes" in their VIX time series. Note that the VIX and the VDAX are weighted indices of implied at the money stock index option volatility. The underlying stock indices are the S&P100 and the DAX, respectively. For a description of the VDAX construction see *Redelberger* (1994). Although the construction of the VDAX is very similar to the one of the

and Trautmann and Beinert (1995) provide evidence of a jump component in German stock and stock index returns. Their empirical results can be interpreted as implicit evidence for a jump component in German stock market volatility. Hence, a jump diffusion framework may provide a more realistic model for a series of implied volatility. The model for the VDAX which is presented in the next section builds upon a proposal by Szimayer and Wagner (1998).

The rest of this paper is organized as follows. In Section II., the volatility models are presented. The extended model specification is characterized by a mean reverting jump diffusion, whereas the alternative specification represents a reduction of the former to the well-know mean reverting diffusion. Section III. outlines the estimation methodology. Section IV. is concerned with an empirical investigation of the German VDAX volatility index based on several alternative model specifications. The question is, whether the more complicated jump model gives a better description of the observed VDAX series. An option pricing application of the model is presented in Section V. The paper ends with a brief conclusion in Section VI.

II. Alternative Volatility Model Specifications

The dynamic behavior of implied volatility is commonly modeled in a stochastic framework. It is assumed that within a frictionless capital market stock market volatility V as a state variable evolves continuously in time. In this section we compare three different models and give a short survey of previous empirical findings.

1. Mean Reverting Jump Diffusion

In the general jump diffusion specification assume that the dynamics of V_t are determined by a stochastic differential equation of the form

(1) $$dV_t = \alpha(L - V_{t-})dt + \sigma V_{t-} dB_t + \kappa V_{t-} dN_t.$$

This equation defines a mean reverting Poisson jump diffusion process (MRJD). The parameters α, L, σ and κ are assumed to be given constants. The function $r(V_t) = \alpha(L - V_t)$ describes a mean reverting volatility com-

VIX, both indices differ with respect to the number of index options used in the calculation, their time to maturity and the underlying weighting scheme (see also *Whaley* (1993)).

ponent with a mean level of L and an instantaneous adjustment rate α. Standard Brownian motion is denoted B_t. The possibility of discontinuous jumps is incorporated via a Poisson process: The stochastic differential dN_t refers to a homogenous Poisson process N_t with intensity $\lambda \geq 0$ per unit time. Accordingly, V_{t-} denotes the state of V_t in advance of a possible Poisson jump event. The processes B_t and N_t in equation (1) are assumed to be stochastically independent. The instantaneous variance of the process-conditional on no Poisson jump event occurring-is denoted by $\sigma^2 > 0$. The parameter κ represents the relative jump height, given a Poisson event occurs.[3]

2. *Mean Reverting Diffusion*

Setting the intensity λ equal to zero in equation (1) yields a model specification where the dynamics of V_t are determined by a stochastic differential equation of the form

$$dV_t = \alpha(L - V_t)dt + \sigma V_t dB_t. \tag{2}$$

This equation defines a mean reverting diffusion process (MRD) with the parameters and the process B_t as given above. The process defining equation (2) itself is a special case of

$$dV_t = \alpha(L - V_t)dt + \sigma V_t^{\gamma} dB_t. \tag{3}$$

Equation (3) was proposed as a starting point for empirical investigations by Chan, Karolyi, Longstaff, and Sanders (1992), denoted CKLS in the following.

3. *A Comparison of the Model Specifications*

In order to model a volatility time series which is reasonable from an economic standpoint, the parameters α, L and σ are assumed to be positive in all model specifications. Furthermore, the variance of changes in volatility increases with the level of volatility. Under models (1), (2) or

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[3] Of course, it would also be possible to assume a stochastic jump height and hence introduce an additional dispersion parameter. The parameter κ would then be the expected jump height (see *Merton* (1976)). Here, the reason for assuming a constant jump height is that in model estimation, using a simpler model is appropriate. Poisson distributed jumps are infrequent events by definition. When estimating not only the average jump height but also its variance, achieving reasonable standard errors for the latter requires large sample sizes.

(3) with $\gamma > 0$ volatility is assumed to be conditionally heteroskedastic. In (3), the degree of heteroskedasticity can be varied by the choice of the parameter γ.

The most relevant difference in the model specifications is the jump component. Taking the expectation of dV_t in the MRJD model (1) we have

$$E(dV_t) = \alpha\left[L - E(V_t)\right]dt + \kappa\lambda E(V_t)dt, \tag{4}$$

as $E(dB_t) = 0$ and $E(dNt) = \lambda dt$. Assuming that the volatility series V_t defined by (1) is stationary and setting the local expected change in V_t equal to zero, equation (4) gives the following solution for the unconditional volatility expectation:[4]

$$E(V_t) = \frac{\alpha}{\alpha - \kappa\lambda} L. \tag{5}$$

For the MRJD model specification, the unconditional expectation (5) exceeds the mean reversion level. For the MRD model specification ($\kappa = \lambda = 0, 0 < \alpha < 1$), the unconditional expectation equals the mean reversion level.

4. *Empirical Studies*

Previous empirical studies of implied stock index volatility by Bühler and Grünbichler (1996), Deutsche Börse AG (1997) and Nagel and Schöbel (1999) are all based on the CKLS-methodology. Bühler and Grünbichler (1996) calculate estimates for the unrestricted model (3) and the square root specification with $\gamma = 1/2$ in (3). The Deutsche Börse AG (1997) uses model (3) for their empirical investigation. Nagel and Schöbel (1999) choose the square root specification as a model of stock market variance.

The overall findings indicate significant parameter estimates including the mean reversion parameter. However, evidence for the square root model is not clear cut. The results of Bühler and Grünbichler (1996) lead to a rejection of the model for the VIX in the period January 1988 to March 1993. Nagel and Schöbel (1999) use the VDAX series in the years

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4 This equation is equivalent to the resulting expectation of a stationary AR(1)-process with heteroskedastic noise term formulated as the discrete-time analogue of the continuous-time mean reverting process defined by (1). The stationarity condition for the specification is: $|1 - \alpha + \kappa\lambda| < 1$ (see e.g. Hamilton (1994, p. 53)). In particular, when assuming $\alpha - \kappa\lambda < 1$ the condition $\alpha > \kappa\lambda$ follows from above. Hence, $0 \leq \kappa\lambda < \alpha < 1 + \kappa\lambda$ is a sufficient stationarity condition.

1992 up to 1995, where the results indicate that the square root specification cannot be rejected at the five percent significance level. A second main finding of the studies is conditional heteroskedasticity in the series which implies a nonzero parameter γ in equation (3). Bühler and Grünbichler (1996) estimate a significant parameter γ of about 1.5 for the VIX. According to the results of Deutsche Börse AG (1997), estimation for the VDAX series from September 1993 to September 1996 yields an estimate of the γ-parameter close to one.[5]

In the remainder of this paper we concentrate on models (1) and (2) which provide nested candidates for the investigation of a jump component in the VDAX series. For comparison purposes, we also provide empirical findings based on the CKLS-methodology.

III. Model Estimation

In this section, we derive a methodology for the estimation of the MRJD model specification defined by equation (1). Using approximate discrete-time versions of the differential equations, the model parameters are estimated consistently with the method of moments approach. The generalized method of moments technique thereby allows to improve the asymptotic efficiency of the estimates and to introduce testable overidentifying restrictions. The estimation methodology is described in detail for the MRJD model specification only, since the estimation procedure for the MRD model specification follows as a simplification without jump component.

1. Discretization of the Model

Following the literature on asset price process estimation (see e.g. Marsh and Rosenfeld (1983) and CKLS (1992)), one may estimate the parameters of the continuous-time model (1) by using a discrete-time approximate specification. We get the following formulation

$$\Delta V_t = V_t - V_{t-1} = \alpha(L - V_{t-1})\Delta t + V_{t-1}\,\sigma\sqrt{\Delta t}\,\varepsilon_t + V_{t-1}\kappa q_t\,, \tag{6}$$

where $\varepsilon_t \sim N(0,1)$ and $q_t \sim Poi(\lambda\Delta t)$ are independent discrete random variables. By a simple transformation we define the relative change in implied volatility as:

[5] Note that the question of whether γ equals a certain nonzero value is of secondary interest.

$$Y_t \equiv \frac{\Delta V_t}{V_{t-1}} = (\alpha L V_{t-1}^{-1} - \alpha + \kappa\lambda)\Delta t + \sigma\sqrt{\Delta t}\,\varepsilon_t + \kappa(q_t - \lambda\Delta t). \tag{7}$$

The estimation methodology is now based on the generalized method of moments (GMM) approach. References are Hansen (1982) and Newey and West (1987b), among others. Textbook treatments are found in Greene (1997) and Hamilton (1994).

2. *On the Choice of the Moment Restrictions*

Naturally, with method of moments estimation, the question arises which moments to choose. In general, we may decide on imposing orthogonality conditions based on the assumption of the uncorrelatedness of certain variables or implement higher order moment restrictions. Apart from that, it still remains a matter of judgement what particular set of conditions is chosen.

With respect to the type of moment conditions, the literature generally suggests to prefer orthogonality conditions and to avoid higher order moments with possibly instable sample properties. Note however, that the GMM approach downweights moment restrictions with higher sample variance. Furthermore, the paper by CKLS is an example for estimating interest rate processes by an overrestricted GMM system with orthogonality conditions. When estimating a volatility process, there is reason to believe that the real world time series dependence structure is more complicated than the one assumed by the linear AR(1)-approximation in equation (6). Empirical results by Franks and Schwartz (1991), for example, have shown that an AR(1) process is a useful but not fully satisfying description of the mean reverting behavior of implied market volatility (see also Bühler and Grünbichler (1996) and Wagner and Szimayer (2001) with results for VIX and VDAX). Hence, it is not surprising that the VDAX estimation results by Nagel and Schöbel (1999) show dependence on the number of sample autocovariances used in the GMM estimation algorithm. This effect should not be present under the true CLKS-model, what is confirmed by the authors in a simulation experiment. Furthermore, note that the CLKS-methodology is by construction not capable of describing the higher order moment behavior which is predicted by model (1). For these reasons, we apply higher order moments in the estimation of our model parameters and test the entire set of moment conditions.[6]

3. *Specification of the Moment Restrictions*

As the volatility processes defined by the differential equations (1) and (2) are characterized by the Markov property,[7] expectations of ΔV_t and Y_t taken conditional on the complete history of the process up to $t-1$ are equal to the expectations taken conditional on the immediate past given by V_{t-1}. From equation (6) it follows for the conditional expected change in volatility

$$E(Y_t\, V_{t-1}|\, V_{t-1}) = (\alpha L + (\kappa\lambda - \alpha)V_{t-1})\Delta t. \tag{8.1}$$

We now derive the conditional uncentered moments of the relative change in volatility given by equation (7). Defining the function

$$f(V_{t-1}) \equiv \alpha L V_{t-1}^{-1} - \alpha$$

it follows (see e.g. Johnson and Kotz (1969) and Johnson, Kotz, and Balakrishnan (1994)):

$$\begin{aligned}
E(Y_t|V_{t-1}) &= \big(\kappa\lambda + f(V_{t-1})\big)\Delta t\\
E(Y_t^2|V_{t-1}) &= (\kappa^2\lambda + \sigma^2)\Delta t + \big(\kappa^2\lambda^2 + 2f(V_{t-1})\kappa\lambda + f(V_{t-1})^2\big)\Delta t^2,\\
E(Y_t^3|V_{t-1}) &= \kappa^3\lambda\Delta t + \big(3\kappa^3\lambda^2 + 3\kappa\lambda\sigma^2 + 3f(V_{t-1})(\kappa\lambda + \sigma^2)\big)\Delta t^2 + O(\Delta t^3),\\
E(Y_t^4|V_{t-1}) &= \kappa^4\lambda\Delta t + \big(7\kappa^4\lambda^2 + 6\kappa^2\lambda\sigma^2 + 3\sigma^4 + 4f(V_{t-1})\kappa^3\lambda\big)\Delta t^2 + O(\Delta t^3),\\
E(Y_t^5|V_{t-1}) &= \kappa^5\lambda\Delta t + \big(15\kappa^5\lambda^2 + 10\kappa^3\lambda\sigma^2 + 5f(V_{t-1})\kappa^4\lambda\big)\Delta t^2 + O(\Delta t^3).
\end{aligned} \tag{8.2-6}$$

The conditional theoretical moments (8.1-6) are given as functions of the parameter vector $\theta = (\alpha,\ L,\ \sigma^2,\ \lambda,\ \kappa)$. Parameter estimation by the method of moments uses the property that the sample moments converge in probability to functions of the model parameters. By virtue of the Slutsky-Theorem, solving for the model parameters yields a consistent estimator of the parameters (see e.g. Greene (1997)). For given empirical

[6] Referring to the issue of choosing a particular set of moment conditions, it can be advantageous to conduct a simulation study in order to assess the performance of a certain estimator under a given moment specification. However, since the overall size of our data set is quite large as compared to those of other studies, we rely on the general asymptotic results for our estimators. The interested reader may refer to *Nagel* and *Schöbel* (1999) who perform a detailed Monte Carlo study focusing on the small sample properties of the GMM-estimators resulting from the CLKS-methodology.

[7] This follows since the volatility driving processes B_t and N_t are both Lévy processes (see *Protter* (1995), p. 238).

observations y_t and $v_t, t = 1, \ldots, T$, and choosing the time interval to be one trading day, $\Delta t = 1$, we can set up the following vector of moment conditions:

$$\mathbf{h}_t(\theta, y_t) = \begin{pmatrix} y_t v_{t-1} - \left[\alpha L + (\kappa\lambda - \alpha) v_{t-1}\right] \\ y_t - \left[\kappa\lambda + f(v_{t-1})\right] \\ y_t^2 - \left[\kappa^2\lambda + \sigma^2 + \kappa^2\lambda^2 + 2f(v_{t-1})\kappa\lambda + f(v_{t-1})^2\right] \\ y_t^3 - \left[\kappa^3\lambda + 3\kappa^3\lambda^2 + 3\kappa\lambda\sigma^2 + 3f(v_{t-1})(\kappa\lambda + \sigma^2)\right] \\ y_t^4 - \left[\kappa^4\lambda + 7\kappa^4\lambda^2 + 6\kappa^2\lambda\sigma^2 + 3\sigma^4 + 4f(v_{t-1})\kappa^3\lambda\right] \\ y_t^5 - \left[\kappa^5\lambda + 15\kappa^5\lambda^2 + 10\kappa^3\lambda\sigma^2 + 5f(v_{t-1})\kappa^4\lambda\right] \end{pmatrix}, \quad t = 1, \ldots, T. \tag{9}$$

Under model (6), ignoring terms of higher order in the theoretical moment equations (8.4-6), the expectation of $\mathbf{h}_t(\theta, y_t)$ is equal to zero. This implies that the vector

$$\mathbf{g}_T(\theta, \mathbf{y}) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{h}_t(\theta, y_t), \tag{10}$$

converges to zero in probability if the empirical observations confirm the set of moment conditions.

4. *Estimation, Asymptotics, and Test Procedures*

The generalized method of moment estimator of the parameter vector θ is given as the minimizer of a quadratic form

$$\hat{\theta} = \arg\min_{\theta} \mathbf{g}_T(\theta, \mathbf{y})' \mathbf{W}\, \mathbf{g}_T(\theta, \mathbf{y}). \tag{11}$$

Consistency of the estimator is given for an arbitrary positive definite weighting matrix **W**. Note that the classical method of moment estimator (MM) follows as a special case of (11) where the number of moment restrictions in $\mathbf{g}_T(\theta, \mathbf{y})$ is equal to the number of parameters in θ. In the just-identified case, the estimator is defined by $\mathbf{g}_T(\hat{\theta}_{MM}, \mathbf{y}) = \mathbf{0}$ and the minimum in (11) exactly equals zero irrespective of the choice of the positive definite weighting matrix **W**. In analogy to the MM-case, it follows for the generalized overidentified case that the asymptotic distribution of the parameter estimate is normal

$$\sqrt{T}(\hat{\theta} - \theta) \xrightarrow{d} N(\mathbf{0};\ \mathbf{V}_T), \tag{12}$$

where the estimate of the covariance matrix is:

$$\hat{\mathbf{V}}_T = \left(\left.\frac{\partial \mathbf{g}_T(\theta, \mathbf{y})}{\partial \theta}\right|_{\theta=\hat{\theta}} \hat{\mathbf{S}}_T^{-1} \left.\frac{\partial \mathbf{g}_T(\theta, \mathbf{y})}{\partial \theta}\right|'_{\theta=\hat{\theta}} \right)^{-1}.$$

Here, $\hat{\mathbf{S}}_T$ denotes an estimator of the asymptotic covariance matrix of the vector of moment conditions times $\sqrt{T}$ which is given by

$$\mathbf{S} = \lim_{T\to\infty} E\left(\sqrt{T}\mathbf{g}_T(\theta, \mathbf{y})\sqrt{T}\mathbf{g}_T(\theta, \mathbf{y})'\right).$$

An autocorrelation and heteroskedasticity consistent estimate of $\mathbf{S}$ which is positive semidefinite by construction is derived in Newey and West (1987a). An optimal selection criterion for the time series lag l used in the estimation of the matrix is e.g. discussed in Newey and West (1994).

It can be shown that the weighting matrix $\mathbf{W}$ in (11) that yields an asymptotically optimal, i.e. smallest, covariance matrix $\mathbf{V}_T$ in (12) is given by $\mathbf{S}^{-1}$ (Hansen (1982), Newey and West (1987b)). When minimizing the quadratic form given by (11), after e.g. starting with the identity matrix, $\mathbf{W} = \mathbf{I}$, the weighting matrix is therefore set $\mathbf{W} = \hat{\mathbf{S}}_T^{-1}$. Commonly, the estimation algorithm continues with iteratively updating the weighting matrix for each new estimate of the parameter vector.

A testable implication of the model specification based on the fit of the overidentified system is (Hansen (1982))

$$Q_T = T\mathbf{g}_T(\hat{\theta}, \mathbf{y})'\hat{\mathbf{S}}_T^{-1}\mathbf{g}_T(\hat{\theta}, \mathbf{y}) \xrightarrow{d} \chi^2(q) \tag{13}$$

with q equal to the number of overidentifying restrictions. Testing the asymptotic significance of single model parameters follows from the asymptotic normality result in (12). Due to the latter, tests of functional forms including subsets of parameters can be performed by the Wald statistic. Under the null hypothesis $\mathbf{f}(\theta) - \mathbf{q} = \mathbf{0}$ it follows (see e.g. Greene (1997))

$$W_T = [\mathbf{f}(\hat{\theta}) - \mathbf{q}]' \left[\left(\frac{\partial \mathbf{f}(\hat{\theta})}{\partial \hat{\theta}'}\right) \hat{\mathbf{V}}_T \left(\frac{\partial \mathbf{f}(\hat{\theta})}{\partial \hat{\theta}'}\right)' \right]^{-1} [\mathbf{f}(\hat{\theta}) - \mathbf{q}] \xrightarrow{d} \chi^2(w), \tag{14}$$

where w is equal to the number of restrictions given by the vector valued function $\mathbf{f}(\theta)$.

IV. Empirical Results

The subject of investigation in the empirical part of this paper is implied stock market volatility measured by the German VDAX. The VDAX is an index measuring volatility implied in call and put option prices where the German stock market index DAX serves as the underlying. The index calculations are based on the assumption that the Black/Scholes option pricing formula is a suitable model for the formation of option prices. Implied volatility is estimated from a subset of liquid at the money options. The contribution of each implied volatility estimate is not subject to an explicit weighting scheme. Instead, weights are determined implicitly by an ordinary least squares regression yielding an aggregate estimate of implied volatility. The index is finally calculated as a time-weighted average of two aggregate implied volatility estimates belonging to two different maturities. This ensures a constant 45 days average time to maturity of the options used in the index calculation (see Redelberger (1994)).

1. The Data Sample

The VDAX sample covers six years of daily data. It includes 1504 observations for the index in the time period beginning on January, 2, 1992 and ending on December, 30, 1997. The empirical frequency distribution of the VDAX series v_t, $t = 0, 1, \ldots, T$, is plotted in Figure 1a). Descriptive statistics indicate that the distribution with a sample mean of 16.68 and a standard deviation of 4.89 is skewed to the right (sample skewness = 1.39) and shows excess kurtosis (sample kurtosis – 3 = 2.59). As can be seen from the histogram, the most frequent class of VDAX observations is approximately centered somewhat below 15, which gives some indication of the reversion level. Figure 1b) plots the empirical distribution of the $T = 1503$ relative VDAX changes $y_t, t = 1, \ldots, T$. It is skewed to the right (sample skewness = 1,51) and exhibits large excess kurtosis (sample kurtosis – 3 = 9.80). The standard deviation of daily relative VDAX-changes equals 0.0437.

Skewness and positive excess kurtosis in Figure 1b) provide evidence for a highly non-normal distribution of the relative VDAX-changes. The hypothesis of normally distributed relative changes can be tested by traditional statistical approaches yielding a rejection of the normality-hypothesis at all conventional confidence levels.

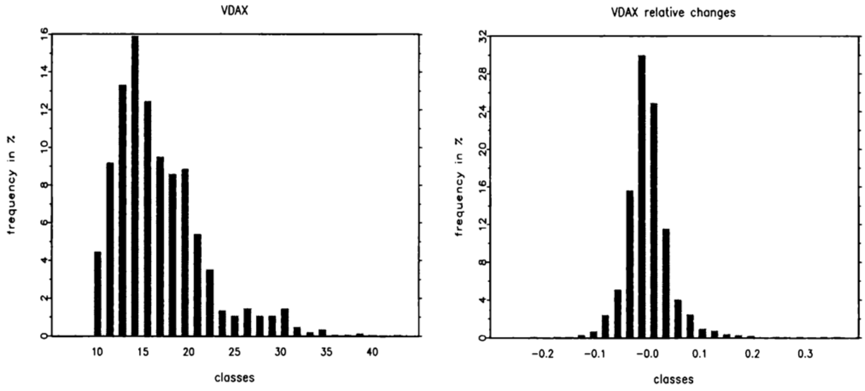

Figure 1: a) Empirical daily VDAX-distribution in the sample period January, 2, 1992 to December, 30, 1997; b) Empirical distribution of the daily relative VDAX-changes ($T = 1503$)

2. Calculation of the Estimates

Following the methodology of Section III., parameter estimation is performed for both, the MRJD and the nested MRD model specification. The minimization problem (11) is solved numerically by application of the Marquardt-algorithm. The estimate of the weighting matrix is updated with each new estimate of the parameter vector. As convergence for the MRJD specification is dependent on the given start solution, MM-estimates based on system (8.1-5) are calculated before continuing with GMM.

Note that the advantage of consistent estimation by the method of moments is that its asymptotic properties hold irrespective of possible heteroskedasticity or autocorrelation in the data. As outlined in Section III., heteroskedasticity and autocorrelation consistent estimation of the covariance matrix preserves optimality of the GMM-estimator. It is therefore a means of robustifying the properties of the GMM-estimator under possible violations of the model assumptions. In the present application, all of the discrete-time model specifications imply that the time series properties of the volatility series are captured by an AR(1)-process (see equation (6)). A possible violation of this assumption does not harm the asymptotic properties of the GMM-estimator (as long as we do not explicitly impose AR(1)-type orthogonality conditions). Hence, when calculating the parameter estimates, the time series lag l for the estimation of the covariance matrix is chosen according to the Newey and West (1994) variable lag selection criterion.

As outlined in Section III., the GMM estimation methodology allows for testing whether the observed data confirm the overall set of moment conditions imposed by the model. A critical test of the overall model restrictions based on Hansen's Q_T statistic (13) has to account for the whole set of testable model restrictions. In the present application, apart from the moment restrictions, an additional restriction is given by the time series assumption of an AR(1)-process. Thus, if the models were correct, a fixed lag of $l = 1$ would be appropriate in order to estimate the true covariance matrix $\mathbf{S}$. Hence, we always choose a fixed lag of one when estimating Hansen's statistic and denote it by $Q_{T|l=1}$. The set of moment conditions (9) provides one overidentifying restriction. Therefore, the $Q_{T|l=1}$-statistic is used for a test of the null hypothesis "$q \leq 1$" under the asymptotic $\chi^2(1)$ distribution.

3. Estimation Results

The parameter estimation results for the MRJD and the MRD model specification are summarized in Table 1, where the standard deviation of the Brownian noise component is consistently estimated by $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$.

Before turning to an interpretation of the results, note that all parameter estimates are significant at least at the 95% confidence level. Par-

Table 1

GMM parameter estimates (T = 1503, system (8.1-6) for the MRJD specification, system (8.1-4) for the MRD specification). Asymptotic t-values based on Newey/West estimates of the standard errors are given in brackets: ** and * denote significance at the 95% and 99% confidence level for a single-sided t-test, respectively. †† denotes significance at the 99% confidence level for a single-sided test under the asymptotic $\chi^2(1)$-distribution**

Parameter	α	L	σ	σ^2	κ	λ	$Q_{T\|l=1}$
MRJD							
Estimate (l=13):	0.0125***	14.21***	0.0356	0.00127***	0.245***	0.00931***	1.22
t-value:	(2.58)	(11.36)		(11.61)	(7.04)	(2.64)	
MRD							
Estimate (l=13):	0.0106**	15.53***	0.0381	0.00138***	–	–	7.66††
t-value:	(2.11)	(13.10)		(10.39)			

ticularly, the basic model restriction $\sigma^2 > 0$ cannot be rejected at the 99% level. The sufficient stationarity condition $0 \leq \kappa\lambda < \alpha < 1 + \kappa\lambda$ holds for the point estimates of the parameters. In Table 2, Wald test statistics for the null hypothesis of equality of each pair of parameters are given. The realizations for the MRJD and the MRD model specification both lead to a rejection of the equality hypotheses at the 95% confidence level. Hence, the assumption of a stationary VDAX series is supported by the estimation results. Particularly, the hypothesis "$\alpha = \kappa\lambda$" can be rejected for the MRJD model at the 95% confidence level. As shown in Section II., equation (5), the implication of the hypothesis would be that the long run unconditional expected volatility is infinite, which seems economically implausible.

Now turn to the results of Table 1 in more detail. Time-series lags of $l = 13$ are chosen for the estimation of the asymptotic covariance matrices. A residual analysis shows that the VDAX time series dependencies are not fully captured by the model. The MRJD as well as the MRD residuals reveal a significant negative first order sample autocorrelation and partial autocorrelation coefficient.

For the MRJD model specification, mean reverting behavior in the VDAX series yields an estimated value of 0.0125 for the α-coefficient. This corresponds to a half-life of 55 trading days.[8] The point estimate of the mean reversion level is 14.21. Hence, any absolute deviation from the

Table 2

Wald test statistics for the GMM parameter estimates. Asymptotic *p*-values for the null hypotheses in brackets. ** and * denote significance at the 95% and 99% confidence level, respectively**

H_0	$\kappa = \lambda = 0$	$\kappa\lambda = 0$	$\alpha = \kappa\lambda$	$\alpha = 1 + \kappa\lambda$
MRJD				
W_T:	$1.59\ 10^2$***	11.2***	4.29**	$4.00\ 10^4$***
p-value:	(0.000)	(0.000)	(0.0384)	(0.000)
MRD			$\alpha = 0$	$\alpha = 1$
W_T:	–	–	4.47**	$3.92\ 10^4$***
p-value:			(0.0345)	(0.000)

[8] Half-life in trading days is computed via the solution to the deterministic differential equation given by the first term in equation (1). This results in a half-life of: $\alpha^{-1} ln 2$.

reversion level will decrease to half of its initial value in about two and a half months of time, neglecting possible intermediate stochastic disturbances.

The economic prediction for κ in the MRJD specification is that the relative jump height should be nonnegative as volatility is a measure derived from squared return deviations. Indeed, the hypothesis "$\kappa = 0$" can be rejected at high confidence levels (see Table 1). The estimated Poisson intensity is highly significant with a value of 0.00931 (asymptotic t-value = 2.64). Hence, in the MRJD model, a shock to implied volatility occurs every 107 trading days on average, then causing an expected relative volatility increase of 24.5%. Both estimates, intensity and jump height are highly significant providing strong evidence for a jump component in the VDAX sample. The Wald test statistic (14) allows a rejection of the hypothesis "The nested MRD model is adequate, κ and λ are equal to zero" at any commonly used confidence level (Table 2, first column). The same holds for the more general hypothesis: "The nested MRD model is adequate, either κ or λ or both are equal to zero" (see Table 2, second column).

The estimate of the mean reversion coefficient in the MRD model specification equals 0.0106 which implies that absolute deviations from the level of 15.53 vanish by half their size within 65 trading days on average. A comparison of the estimation results of the MRJD and the MRD model specification in Table 1 shows that the scale parameter estimate is highly significant in both cases. Obviously, Brownian noise is the dominating part in driving implied volatility. The mean reversion coefficient estimate implies an approximately identical half-life for both model specifications. As shown in Section II., a stationary MRJD volatility time series has an unconditional expectation which is larger than the mean reversion level. Using point estimates in equation (5), the unconditional expectation for the MRJD model is equal to 1.22 times the estimated mean reversion level of 14.21. This yields a value of 17.38. On the other hand, expected volatility in the MRD model specification is known to be equal to the mean reversion level which is 15.53. Recall that the VDAX sample average is 16.68, which corresponds to the expectation of a Brownian noise model without mean reversion and jumps. Obviously, deriving a long run volatility expectation highly depends on the underlying model assumptions.

The overall fit of the MRJD and the MRD model specification shows strikingly different results. While the Q-statistic is not significantly different from zero under the MRJD specification, the MRD specification has to be rejected at the 99% confidence level. We conclude that the moment conditions of the mean reverting diffusion model are not supported by the empirical VDAX observations.[9] For the given sample, the extended model provides an alternative which cannot be rejected at commonly used significance levels.

4. Results for Alternative Estimation Approaches

In this section we first investigate the sensitivity of our estimation results with respect to the overidentifying restriction. In order to provide a basis for comparisons, we additionally estimate the MRD model specification based on the CKLS-methodology and test its moment restrictions for the given VDAX sample.

Table 3 shows the estimation results based on a reduced system of moment restrictions. The reduction is achieved by dropping the highest order moment, yielding a just-identified non-linear system that puts equal weight on all moment conditions. The resulting MM-estimator has lower efficiency than the GMM-estimator. Nevertheless, the estimation results in Table 3 show that all parameter estimates stay significant at the 95% confidence level. The MM-estimates of the reversion rate are nearly unchanged while the estimates of the level and the standard deviation provide higher values than in the GMM case. The estimate of the Poisson intensity is lower and the estimate of the relative jump height is larger when compared to the GMM results.

The CKLS-methodology derives moment restrictions for the expectation and the variance of discrete changes in volatility. Additionally, it imposes two orthogonality conditions. The resulting estimates for the VDAX sample are given in Table 4. Apart from the MRD model specification according to equation (2), estimation results are given for the square root specification with $\gamma = {}^1/_2$ in model (3). In both cases, all estimates are highly significant. Nevertheless, the Q-statistic leads to a rejection of both model specifications for the given sample (95% and 99% confidence level.). As found in previous empirical studies, the square root specification shows an inferior fit when compared to model specification (2).

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[9] This main result persists when the moment conditions (8.1-3, 8.5) instead of (8.1-4) are chosen for estimating the parameters of the MRD model specification. Calculating $Q_{T|l=1}$ leads to a rejection of the model at the 97.5% confidence level in this case while the parameter estimates show lower t-statistics.

Table 3

MM parameter estimates (T = 1503, system (8.1-5) for the MRJD model specification, system (8.1-3) for the MRD model specification). Asymptotic t-values based on Newey/West estimates of the standard errors are given in brackets. ** and * denote significance at the 95% and 99% confidence level for a single-sided t-test, respectively**

Parameter	α	L	σ	σ^2	κ	λ	$Q_{T\|l=1}$
MRJD							
Estimate ($l = 13$):	0.0123***	15.32***	0.0381	0.00146***	0.284***	0.00554**	–
t-value:	(2.49)	(11.89)		(8.21)	(6.81)	(1.84)	
MRD							
Estimate ($l = 13$):	0.0107**	17.56***	0.0436	0.00190***	–	–	–
t-value:	(2.15)	(9.49)		(8.49)			

Table 4

GMM parameter estimates for the MRD model (3) with constant γ parameter under the CKLS moment restrictions (T = 1503). Asymptotic t-values based on Newey/West estimates of the standard errors are given in brackets. * denotes significance at the 99% confidence level for a single-sided t-test. † and †† denote significance at the 95% and 99% confidence level for a single-sided test under the asymptotic $\chi^2(1)$-distribution, respectively**

Parameter	α	L	σ	σ^2	$Q_{T\|l=1}$
MRD					
Estimate ($l = 24$):	0.0167***	16.59***	0.0430	0.00185***	4.59†
t-value:	(3.79)	(16.64)		(8.01)	
MRD with $\gamma = {}^1/_2$					
Estimate ($l = 25$):	0.0176***	16.35***	0.171	0.0291***	8.51††
t-value:	(3.98)	(17.61)		(7.33)	

V. Option Pricing Under Risk Neutrality

As pointed out in the introduction, a market for volatility may serve the needs of market participants to hedge against changes in the level of implied market volatility. This can be achieved by the introduction of volatility derivatives such as futures and options. The aim of a theoretical framework will be to explain the prices of the derivative securities assuming a model which captures the dynamics of the underlying. In this section we present an option pricing application based on the models discussed above. For an overview on option pricing and model estimation also refer to Campbell, Lo, and MacKinlay (1997).

1. *Preliminaries*

Unfortunately, pricing options on an implied volatility series has several features that do not allow straightforward application of standard option pricing techniques. The standard theory of option pricing is based on arbitrage arguments and hence critically relies on the assumption of an underlying asset which can be traded in a frictionless market. However, the VDAX is not tradable.[10] Even if it was, the empirical results presented above demonstrate that the assumption of frictionless trading in the underlying will be violated due to discontinuities. Furthermore, the convenient assumption that jump risk is unsystematic and hence unpriced in a CAPM-world (see Merton (1976)) is not appropriate in the given context. Plenty of evidence in the finance literature shows that stock market volatility is negatively correlated with the level of market prices (so called "leverage-effect"). Consequently we cannot assume that jumps in VDAX volatility are unsystematic.

While payoff replication breaks down, it is still possible to price the volatility contingent claim based on its payoff distribution. A risk neutral pricing framework with a non-tradable underlying is therefore most appropriate to make inferences about VDAX option values (see also the approach of Günbichler and Longstaff (1996)). In a risk-neutral valuation framework, the following standard pricing formulas for European calls and puts can be applied:

$$C_t = E\big(\exp\big(-r(T-t)\big)F_T|\Phi_t\big), \qquad F_T = \max\{V_T - X;\ 0\}, \tag{15}$$

$$P_t = E\big(\exp\big(-r(T-t)\big)F_T|\Phi_t\big), \qquad F_T = \max\{X - V_T;\ 0\}. \tag{16}$$

In the above formulas, a flat term structure is assumed where the constant discount rate under continuous compounding is given by r. The

[10] The VOLAX as a future on the VDAX was an attempt to provide a tradable instrument (see Deutsche Börse AG (1997)). In principal, the *Cox/Ingersoll/Ross* (1985a) concept of risk neutral valuation with a non-tradable underlying makes the introduction of a risk premium necessary and hence does not allow preference-free pricing.

symbol Φ_t represents available pricing information at time t including the initial state of the underlying given by V_t. X is the exercise price of the option. V_T denotes the terminal value of the underlying[11] and the time to maturity is given by $(T-t)$.

2. *Simulated Call Option Prices*

With our parameter estimates of Section IV., it is possible to simulate European VDAX call option prices as defined by equation (15).[12] Available pricing information Φ_t is given by the assumed parameterization of the VDAX process and the start value V_t. Performing m independent Monte Carlo runs it follows from (15)

$$\hat{C}_t = \exp\bigl(-r(T-t)\bigr) \cdot \frac{1}{m} \sum_{i=1}^{m} \bigl(F_{i,T} \big| V_t, \hat{\theta}_t\bigr). \tag{17}$$

The standard errors of the above simulated option price depends on the precision of the parameter estimates and the number of simulation runs. Ignoring estimation risk for the model parameters, for large m, the Central Limit Theorem implies:

$$\sqrt{m}(\hat{C} - C) \xrightarrow{d} N(0, \sigma_F^2) \cdot$$

Hence, the standard error of the simulated option price (17) can be estimated by:

$$\frac{\hat{\sigma}_F}{\sqrt{m}} = \frac{1}{m} \sqrt{\sum_{i=1}^{m} \bigl(F_{i,T} - \bar{F}_T\bigr)^2}\,.$$

As a result of Section IV., we point out that the assumption of a MRJD versus a MRD model specification is relevant not only because of the possibility of jumps in the underlying series, but also because it has an essential effect on the estimation results. Price simulations are possible

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11 More precisely, it denotes the terminal value of the underlying under risk-neutrality. However, when the underlying variable is not a traded asset which is held for investment purposes, this distinction does not apply (see also *Hull* (1997), Chapter 13).

12 For the sake of brevity, put option prices are not dealt with explicitly, they follow analogously from equation (16). Note however that since the estimated jump component with positive sign corresponds to an asymmetry in the VDAX-distribution, the pricing implications under a jump model will be different for put options.

under many different scenarios and model assumptions. Based on our empirical findings, we choose the GMM-estimates from Table 1 for the MRJD model specification. In order to compare the results with a standard method of inference, we choose the CKLS-estimates for simulating European call prices under the MRD model specification (Table 4, $\gamma = 1$). The sample standard deviation of relative VDAX changes y_t is used in order to calculate hypothetical option prices under the Black/Scholes model. The continuously compounded annual rate in all models is $r = 0.03$.

The pricing results for five different strikes and three maturities are summarized in Tables 5 and 6. As an initial VDAX-quotation $V_t = 14$ is chosen in Table 5, $V_t = 20$ is chosen in Table 6. A graphical overview showing the corresponding Black/Scholes-, MRD- and MRJD-prices is given in Figure 2.

The results indicate that call option prices under the three model assumptions deviate for all but very short maturities. Under a mean reverting process, call option prices generally show significant deviations from Black/Scholes-prices. Call options with longer maturities have higher prices assuming a geometric Brownian motion. Under an initial VDAX which is lower than the estimated mean reversion level, Black/Scholes results in lower prices for in-the-money calls with short to mid maturities. Under an initial VDAX which is above the estimated mean reversion level, Table 6 and Figure 2 show that the Black/Scholes model yields relatively high option prices for all assumed strikes and maturities.

When characterizing differences between MRD and MRJD prices, Table 5 and 6 show that the MRJD model specification generally yields higher option prices. The MRD model specification especially yields lower prices for out-of-the-money options and options with long maturities. Under an initial VDAX of 14, Figure 2 illustrates that MRJD prices are characterized by a surface which mostly lies in-between Black/Scholes and MRD prices.

A comparison of the left and the right column in Figure 2 shows that under mean reversion, option prices for longer maturities are much less dependent on the initial VDAX-quotation than those calculated for shorter maturities. A comparison of the prices for options maturing in 240 trading days given in Table 5 and 6 indicates that under the MRJD specification the price differences depending on an assumed initial $V_t = 14$ versus $V_t = 20$ are larger then under the MRD specification. In-

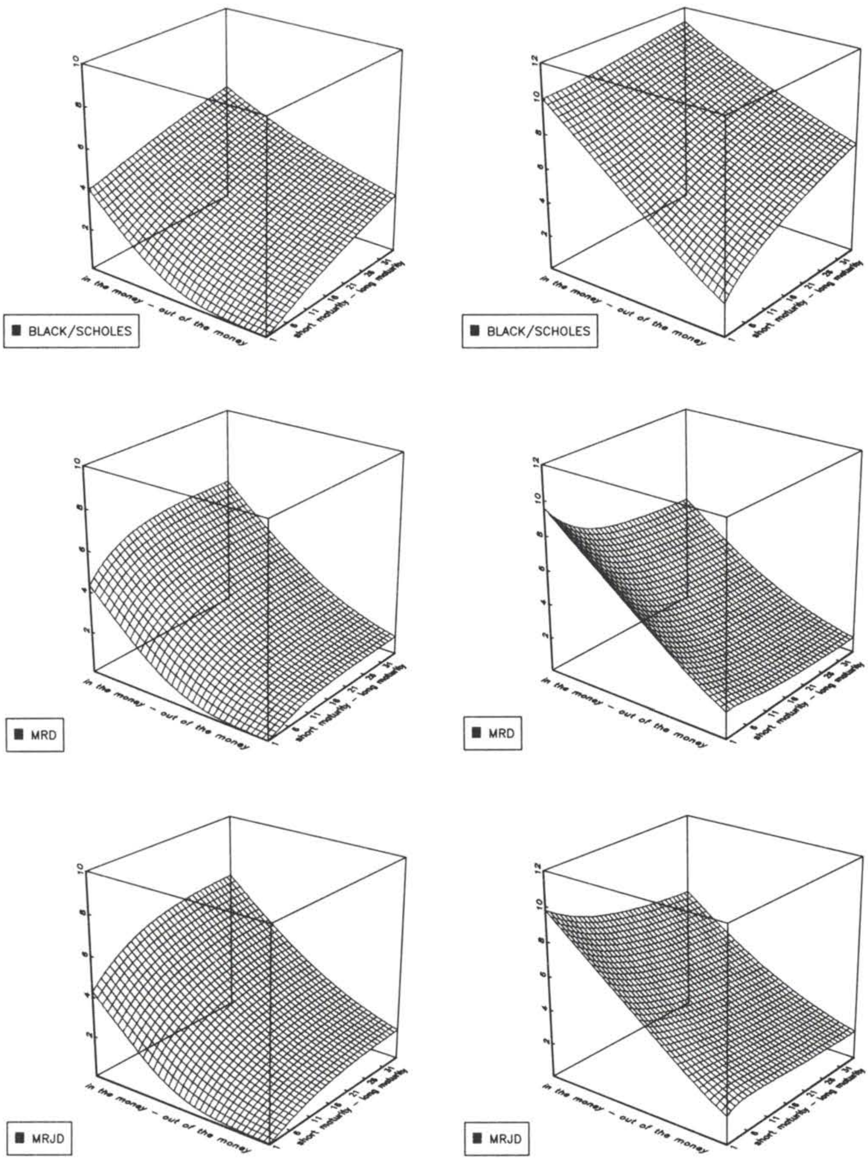

Figure 2: From top to bottom: a) Black/Scholes, b) MRD and c) MRJD call option prices. Left hand side: $V_t = 14$. *Right hand side:* $V_t = 20$. *(Black/Scholes:* $\hat{\sigma} = 0.0437$, *MRD and MRJD simulated with 100.000 runs and parameters from Table 1 and 4; strikes* $\in \{10,\ 10.25,\ \ldots,\ 18.75\}$, *maturities* $\in \{8,\ 16,\ \ldots,\ 280$ *[trading days]*$\}$*)*

Table 5

Black/Scholes (BS)-, MRD- and MRJD-call option prices with $V_t = 14$. (Black/Scholes: $\hat{\sigma} = 0.0437$, MRD and MRJD prices are based on 500.000 Monte Carlo runs with standard errors given in parenthesis. Parameter estimates are from Table 1 and Table 4, respectively)

$(T-t)$:		80			160			240		
		BS	MRD	MRJD	BS	MRD	MRJD	BS	MRD	MRJD
X:	12	3.23 –	4.00 (0.0049)	4.06 (0.0056)	4.05 –	4.43 (0.0054)	4.80 (0.0067)	4.69 –	4.52 (0.0055)	5.10 (0.0072)
(V_t)	14	2.23 –	2.45 (0.0043)	2.59 (0.0050)	3.15 –	2.84 (0.0048)	3.28 (0.0061)	3.86 –	2.93 (0.0050)	3.57 (0.0066)
	16	1.50 –	1.36 (0.0034)	1.55 (0.0042)	2.45 –	1.68 (0.0040)	2.14 (0.0054)	3.19 –	1.75 (0.0042)	2.39 (0.0059)
	18	1.00 –	0.69 (0.0026)	0.89 (0.0033)	1.91 –	0.93 (0.0031)	1.36 (0.0045)	2.64 –	0.98 (0.0033)	1.57 (0.0050)
	20	0.66 –	0.34 (0.0018)	0.51 (0.0026)	1.49 –	0.49 (0.0023)	0.86 (0.0037)	2.20 –	0.53 (0.0025)	1.02 (0.0042)

Table 6

Black/Scholes (BS)-, MRD- and MRJD-call option prices with $V_t = 20$. (Black/Scholes: $\hat{\sigma} = 0.0437$, MRD and MRJD prices are based on 500.000 Monte Carlo runs with standard errors given in parenthesis. Parameter estimates are from Table 1 and Table 4, respectively)

$(T-t)$:		80			160			240		
		BS	MRD	MRJD	BS	MRD	MRJD	BS	MRD	MRJD
X:	12	8.37 –	5.48 (0.0058)	6.52 (0.0073)	8.97 –	4.82 (0.0057)	5.88 (0.0077)	9.55 –	4.62 (0.0056)	5.58 (0.0076)
	14	6.74 –	3.75 (0.0054)	4.77 (0.0070)	7.59 –	3.18 (0.0052)	4.25 (0.0072)	8.32 –	3.02 (0.0050)	3.99 (0.0071)
	16	5.32 –	2.37 (0.0047)	3.31 (0.0063)	6.39 –	1.95 (0.0044)	2.95 (0.0065)	7.25 –	1.83 (0.0043)	2.75 (0.0064)
	18	4.14 –	1.40 (0.0038)	2.21 (0.0055)	5.37 –	1.12 (0.0035)	2.00 (0.0057)	6.32 –	1.04 (0.0034)	1.85 (0.0056)
(V_t)	20	3.18 –	0.79 (0.0030)	1.43 (0.0046)	4.51 –	0.62 (0.0027)	1.33 (0.0049)	5.51 –	0.56 (0.0026)	1.24 (0.0048)

tuition suggests that this is due to the fact that under the MRJD model, the magnitude of a possible jump depends on the most recent level of the VDAX.

3. Pricing Implications of the Jump Component

In order to make inferences about the effects of the jump component in pricing options, we perform the price simulations based on a set of parameters which imply identical first and second moments of the simulated payoff distributions. Under this condition, it is possible to isolate the effects of different higher order moments of the MRJD versus MRD payoff distribution.

More precisely, we require that the unconditional VDAX-expectation $E(V_t)$ and the unconditional variance of the relative VDAX-changes Y_t are identical under the MRD and MRJD estimates. This condition holds under MM-estimation.[13] Equality of the unconditional expectation requires with (5) that the equation

$$\hat{L}_{MM\_MRD} = \left(\frac{\hat{\alpha}}{\hat{\alpha} - \hat{\kappa}\hat{\lambda}}\right)_{MM\_MRJD} \hat{L}_{MM\_MRJD} \tag{18}$$

holds. Additionally, the equation for the unconditional variance

$$\hat{\sigma}^2_{MM\_MRD} = \hat{\sigma}^2_{MM\_MRJD} + \left(\hat{\kappa}^2\hat{\lambda}\right)_{MM\_MRJD} \tag{19}$$

holds as well. Under MM-estimation, both model specifications yield estimates which imply identical estimates of the expectation and the standard deviation of the VDAX series. As a consequence, jump specific price differences can be examined in isolation.

European call option prices are now simulated based on the MM-estimates from Table 3. Table 7 shows MRJD and MRD prices with their estimated standard errors. The relative pricing error of the MRD versus the MRJD model specification, $(\hat{C}_t^{MM\_MRD} - \hat{C}_t^{MM\_MRJD})/\hat{C}_t^{MM\_MRJD}$, is given in percentage terms. Table 7 reports results for the case $V_t = 14$ where in-the-money, at-the-money and out-of-the-money call prices can be examined.

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[13] A proof of this statement follows from solving the MM-estimation equations (8.1-3) with $\kappa = \lambda = 0$ for the MRD specification and solving equations (8.1-5) for the MRJD specification.

Table 7

MRD and MRJD call option prices based on MM-estimates with relative price differences in percent. # denotes non-overlapping 1.96-sigma intervals for the simulated prices. ($V_t = 14$, MRD and MRJD prices are based on 500.000 Monte Carlo runs with standard errors given in parenthesis. Parameter estimates are from Table 3.)

$(T-t)$:		80			160			240		
		MRD	MRJD	Δ%	MRD	MRJD	Δ%	MRD	MRJD	Δ%
X:	12	4.23 (0.0057)	4.21 (0.0058)	0.6#	5.03 (0.0068)	4.99 (0.0069)	0.6#	5.33 (0.0072)	5.28 (0.0073)	0.9#
(V_t)	14	2.75 (0.0051)	2.71 (0.0052)	1.4#	3.49 (0.0062)	3.45 (0.0063)	1.2#	3.78 (0.0067)	3.72 (0.0067)	1.5#
	16	1.67 (0.0042)	1.65 (0.0043)	1.7#	2.31 (0.0055)	2.28 (0.0055)	1.6#	2.57 (0.0059)	2.51 (0.0060)	2.0#
	18	0.97 (0.0033)	0.96 (0.0035)	0.8	1.48 (0.0046)	1.46 (0.0047)	1.5	1.69 (0.0051)	1.66 (0.0052)	2.0#
	20	0.54 (0.0025)	0.55 (0.0027)	-1.7	0.93 (0.0038)	0.93 (0.0039)	0.5	1.10 (0.0042)	1.08 (0.0043)	1.4

The simulation results show that the estimated standard errors are very close to each other as the variance of the payoff distribution is identical. The jump simulation has slightly higher standard errors. Relative to the MRJD specification, the MRD specification overprices in-the-money call options and call options with long maturities. The MRJD specification yields higher prices for short maturity far out-of-the-money calls, although they are not (yet) significant for a strike of 20 and 80 trading days maturity. MRJD and MRD prices differ insignificantly for far out-of the-money calls with a strike of 20 (see Table 7).

VI. Conclusion

Any volatility model represents a simplified framework which approximates the time-varying behavior of real world market volatility. There is strong evidence in the literature that the dynamic behavior of implied stock index volatility deviates from a process of geometric Brownian motion. In order to obtain a realistic model of the VDAX-series, we show that both mean reversion and the possibility of jumps are important characteristics. Our model extends previously suggested volatility models by a homogeneous Poisson jump component. As demonstrated by the empirical investigation, the extended model provides highly significant evidence of a jump component. The empirical properties of changes in the VDAX-series are not adequately captured by the moment conditions derived from the standard mean reverting diffusion model. The more general model specification with jumps provides a superior empirical fit where the applied test statistic accounts for the increased number of parameters. Modeling a series of implied volatility such as the VDAX is important for risk management and option pricing applications. Focusing on simulated European call option prices, we demonstrate that not only model selection but also the jump component itself has a significant pricing impact.

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Summary

Alternative Model Specifications for Implied Volatility Measured by the German VDAX

In this paper, two nested model specifications for the stochastic behavior of the German stock market volatility index VDAX are compared based on a sample of index observations. Following the literature, the well-known mean reverting diffusion model serves as the standard model specification. The second model specification is an extension which allows for discontinuous changes in the series. The estimation results for the VDAX indicate that the empirical observations do not confirm the moment restrictions given by the standard model. While, at the given confidence level, this yields to a rejection of the mean reverting diffusion model, the extended specification cannot be rejected providing significant evidence of a positive jump component. An application of the mean reverting jump diffusion model is given in a risk-neutral option pricing framework. Simulated option prices reveal economically and statistically significant price differences not only depending on the choice of the model specification but also due to the consideration of the jump component itself. (JEL C13, C15, C22, G13)

Zusammenfassung

Zur Modellierung des VDAX-Volatilitätsindexes

In dieser Arbeit werden zwei verwandte Ansätze zur Modellierung des VDAX-Volatilitätsindexes anhand einer historischen Zeitreihe verglichen. Der Standardansatz ist das weitverbreitete „Mean Reverting Diffusion"-Modell. Die Alternative besteht in einer Erweiterung, welche die Möglichkeit von Sprüngen in der Zeitreihe berücksichtigt. Die empirischen Ergebnisse zeigen, daß die Momentenrestriktionen des Standardmodells abgelehnt werden müssen, während das erweiterte Modell bei gegebenem Konfidenzniveau nicht verworfen werden kann. Letztere Modellspezifikation liefert den empirischen Beleg für eine statistisch signifikante, positive Sprungkomponente in der VDAX-Zeitreihe. Eine mögliche Anwendung des Diffusionsmodells mit Mean Reversion und Sprüngen wird anschließend im Bereich der risikoneutralen Optionsbewertung aufgezeigt. Die Simulation der Preise von europäischen Kaufoptionen belegt ökonomisch und statistisch signifikante Preisdifferenzen, die sowohl auf die Wahl der Modellspezifikation als auch explizit auf die Berücksichtigung der Sprungkomponente zurückzuführen sind.

Résumé

Modèles alternatifs pour la volatilité mesurée par l'indice allemand VDAX

Dans cet article, deux modèles apparentés de l'indice de volatilité sont comparés. L'approche standard est celle du modèle de la diffusion «mean reverting». L'approche alternative est celle d'un élargissement du modèle qui considère la possibilité de discontinuité dans la série chronologique. Les résultats empiriques montrent que des restrictions momentanées du modèle standard doivent être rejetées alors que le modèle élargi ne peut pas être écarté pour un niveau de confiance donné. Cette spécification du modèle prouve l'existence d'un composant de discontinuité statistiquement significatif dans la série chronologique VDAX. On montre ensuite l'utilisation possible du modèle de diffusion avec la «mean reversion» et les discontinuités lorsqu'on évalue des options neutres par rapport aux risques. La simulation des prix d'options d'achat européennes révèle des différences de prix statistiquement significatives qui s'expliquent par le choix de la spécification du modèle et par le composant explicite de discontinuité.