Pricing of Caps and Floors
A Simplified Approach

By Hans H. Hubbes, London

Introduction

The efficient management of interest rate risk is the dominant challenge in the current environment of high interest rate volatility. A broad spectrum of products is available today to exposed parties such as borrowers, issuers and investors. Financial futures, first introduced in 1977, are often referred to as the standard hedging tool. Professional traders have made the futures market highly successful. The way to use these instruments in controlling interest rate risk is, however, not always evident. Experience, attention and time are required to effectively structure and monitor a futures hedge. Among corporations, only a few have developed the sophistication necessary to use futures as a hedging vehicle for interest rate exposure.

Financial institutions have used futures to innovate the forward rate agreements as tailor-made hedges for clients. These instruments allow the borrower to fix rates over two to ten year periods while eliminating basis risk and margin requirements.

Also, interest rate swaps have progressively gained popularity among corporate clients over the last five years. They enable floating rate borrowers to gain access to fixed rates, while fixed rate borrowers can tap the floating rate market.

Options on financial futures were introduced as exchange traded instruments about four years ago. Trading of options on Eurodollar contracts began in 1985.

Caps and floors are the most recent innovations in the financial markets. Cap agreements were settled for the first time about two years ago. In the then prevailing high interest rate environment this was a logical progression. While interest rates came down over the last two years, financial institutions innovated the floor agreement as the counterpart to the cap agreement. Caps and floors became a powerful hedging tool adapted to the individual need of the client. In this paper we want to look closely at these instruments to improve our understanding. A simplified, but operational approach to the pricing of caps and floors will be discussed in the context of hedging an interest rate option.

I. Definition of Caps and Floors

Caps and floors are simply options on interest rates. Borrowers use cap agreements to put a ceiling on floating interest rate costs. This can result in important cost reductions compared with fixed rate alternatives. When the term structure is upward sloping, as it often is in the US-Dollar market, borrowing short term is more attractive than long term fixed rate alternatives, such as bonds or interest rate swaps, which are priced off the treasury bond yield curve. The fixed rate borrower incurs no risk when interest rates rise but he also enjoys no benefit in an environment of declining interest rates. The holder of a cap is protected against the increase in interest rates as is also the fixed rate borrower. The potential to benefit from lower interest rates is, however, maintained for the cap holder while the fixed rate borrower foregoes this potential. Cap agreements cater for the short term borrower, floor agreements cater for the short term lender. Floors are attractive to clients who want to obtain a minimum return in a low interest rate environment.

The beauty about caps and floors lie in their simplicity and practicability. There is a strong possibility that these new instruments will outperform money-market futures and options. Today there is increasing trade in caps and floors. Quotes are given as upfront fees in Basis Points (BP) of the principal. Expiry periods range from two to twelve years; six month or three month LIBOR is often used as a reference interest rate. Underlying principals rank in the order of some ten million US-Dollars.

Although upfront fees for caps and floors seem to be considerable, they may easily prove to be cost efficient hedging vehicles compared with futures and options on futures. Futures do not offer a cost free hedge. They just incur costs in establishing a position through initial margin and variation margin when the market turns against the futures position.

There is no particular reason for trading caps and floors on the basis of a one-off fee. Rather, since caps and floors are essentially an insurance against unfavourable interest rate movements, one can think of paying a regular “insurance premium” to the writer of the option for the time the agreement holds. Caps and floors in currencies other than US-Dollars are still a rarity but are steadily becoming more common with easier access to the interbank interest rates in these currencies. DM caps and floors are already trading regularly.

II. How Caps and Floors Work

Similar to options on any security, caps and floors give a right to the buyer of the option and let the seller incur an obligation. The cap is an option on a borrowing rate. The buyer has the right to borrow at a fixed rate, which is termed the "exercise rate" of the cap. The holder of the cap will exercise his option to borrow at the cap rate when market interest rates are above the cap rate. In this case the caps are "in the money", using option terminology. If market interest rates are below the cap rate, the cap has no value to the owner, because he can borrow at the lower market interest rate. Similarly, the floor is an option on a lending rate. The holder of the floor has the right to lend at the "exercise rate" of the floor agreement. The floor is "in the money" when market interest rates are below the floor interest rate.

Interest rate options are related to swap agreements. No principal is exchanged in general. Settlement takes place on the delivery date, which is fixed in the floor or cap agreement. On this date the difference of funding or lending is exchanged. The terms of the cap or floor agreement state the reference interest rate (LIBOR, Prime, Commercial Paper) and the way in which the difference between the benchmark and the exercise interest rate is calculated.

The writer of the cap or floor assumes the interest rate risk of the buyer of the option. He provides, in fact, a service for which he is charging a fee. The holder of an "in the money" cap may, as a practical matter, borrow from the seller of the option, who, in turn, would have to fund himself at the then prevailing higher market interest rate. The seller of the cap then incurs funding costs, which are higher than his interest income. He runs a loosing position for which he should have established a hedge.

III. Evaluation of Interest Options

We have placed the pricing of interest rate options into a *Cox-Rubinstein* option pricing framework.[1] The Cox-Rubinstein option pricing model is a simple and illustrative concept. Following the Cox-Rubinstein method one can avoid the often difficult solution of complex differential equations. The method is mainly used to price American style options, which can be exer-

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[1] *Cox, John C.; Rubinstein,* Mark, Options Market, Englewood Cliffs, 1985.

cised at any time. The flexibility which is achieved using the Cox-Rubinstein method is paid for with a somewhat longer computation time compared with other models such as *Black-Scholes*[2], or *Fischer-Black*.[3] The latter gives closed form solutions for the pricing of options on equity and futures contracts, but they are not compatible with interest rate options as will be shown later. In developing a pricing model for interest rate options we have pursued a three step approach, which will be discussed in detail. Any option pricing model is good as long as it achieves a degree of accuracy in indicating the costs of hedging a written option and derives a hedge ratio. Following the approach of Cox-Rubinstein one then has to build the model on the equivalent portfolio. The equivalent portfolio consisting of underlying security and riskless deposit duplicates the cash flow pattern of the option. Accordingly, the equivalent portfolio indicates the riskless hedge strategy. If the perfect hedge can be constructed, the funds invested should earn the riskless rate of interest. The definition of the equivalent portfolio is the first step in the development of the option pricing model.

The second step is to agree upon a process in which the prices of the underlying security move. This is the stochastic process of interest rates which is not evident. As a first approach, we have assumed that proportional changes of interest rates are normally distributed around a mean equal to zero. Only a single measure, the interest rate volatility, is used as input into the model to describe interest rate risk. It is understood that interest rate volatility is equal to the standard deviation of the underlying density function of proportional changes. The market is using this measure in communicating historical or expected behaviour of interest rates. In using volatility only, we essentially adopt a stochastic process of interest rates as also assumed for share prices and quotes of futures contracts in the models of Black-Scholes and Fischer Black.

Other authors[4] have argued the simple approach used here to describe the stochastic process of interest rates. It has been shown that the volatility of interest rates depends on the respective level of interest rates.[5] This feature

[2] *Black, Fischer; Scholes, Myron,* The Pricing of Options on Corporate Liabilities, in: Journal of Political Economy, 1973, S. 637 - 654.

[3] *Black, Fischer,* The Pricing of Commodity Contracts, in: Journal of Financial Economics, 1976, S. 167 - 179.

[4] *Mobbs, Stephen,* Evaluating Interest Rate Caps, in: International Bond Market Research, C.S.F.B., 1985; *Schaefer, Stephen M.; Schwartz, Eduardo S.,* Time-Dependent Variance and the Pricing of Bond Options, London Business Scholl, University of British Columbia, 1985.

[5] *Mobbs, Stephen,* Evaluating Interest Rate Caps, in: International Bond Market Research, C.S.F.B., 1985.

remains unconsidered in our model. Unfortunately, a more sophisticated description of the way in which interest rates move requires additional input which is to a certain extent subjective. The marginal benefit from more complicated models is questionable.

As a final step, one has to agree upon a method to define the current environment in which the option is written. Caps and floors are multi-period European type options, i.e. exercise in between the termed delivery dates is excluded. As such, interest rate options are options on expected short term interest rates. From this it is evident that a certain amount of subjective input is required in order to translate current conditions into expectations. Forward rates are easily calculated from the prevailing yield curve. One should however keep in mind that resulting forward interest rates are generally not a good indicator of future interest rates because the shape of the yield curve, from which they are calculated, is subject to various influences such as market supply and demand. For interest rate options running longer than five years one is generally obliged to extrapolate the interbank interest rates because quotes beyond this schedule are no longer available. One has to be cautious with these estimates because they are sensitive factors in the pricing model.

IV. The Equivalent Portfolio

When looking upon caps and floors as options, one has to be aware of the nature of the underlying security. We are dealing with swap agreements which do not involve the exchange of principal but only the compensation for a difference in quotes of interest rates. This indicates that we can compare caps and floors with options on futures contracts. The holder of an option on a futures contract assumes, through exercise, a position in a futures contract and not in a cash position. The value of a futures contract is equal only to the difference between the initial quote of the contract, when the position has been established, and the quote of the day. As for a call option on a futures struck at 100 and exercised when the contract is quoted at 105, results in a futures position of value equal to five points ($= 105 - 100$) and not 105 of whatever unit. One can also think of an option on a futures contract as an option struck at US-Dollars 0 on the value of the futures contract (not to be confused with the quote of the futures contract).

Similar considerations apply to caps and floors and their underlying contracts. The contract which meets the requirements within an equivalent portfolio with caps and floors can be defined as follows:

$$F(E, S) = \frac{1 + E}{1 + S} \times 100 \tag{1}$$

with E = Exercise Interest Rate
S = Market Interest Rate
F = Present Value of the Exercise Rate

In order to illustrate the particular features of this contract let us assume that interest rates branch over one period into two outcomes of equal probability as depicted in fig. 1.

Fig. 1. "Interest rate tree" with interest rates initially at 7% and after 1 period either at 5% or at 10%. We assume that both outcomes have equal probability of 1/2.

$$7\% \begin{array}{l} \nearrow \ 5\% \text{ (probability 1/2)} \\ \searrow \ 10\% \text{ (probability 1/2)} \end{array}$$

time: $t = t_o$ $\qquad t = T$

with t_o: current date
T: end of period

The exercise interest rate is set at 7% in this example. The initial market interest rate of 7% is consequently equal to the exercise interest rate. Using equation (1), the "interest rate tree" of fig. 1 can be translated into a quote tree as shown in fig. 2.

Fig. 2. The quotes of the contract defined in Eq. 1 given the interest rate branch in Fig. 1. We assume that one period is equal to 1 year.

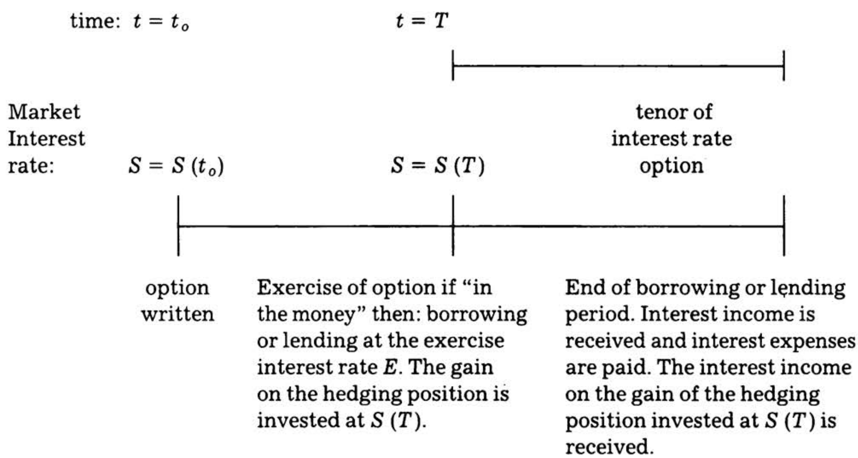
$$100 \begin{array}{l} \nearrow \ 101.9 \\ \searrow \ 97.3 \end{array}$$

time: $t = t_o$ $\qquad t = T$

with t_o: current date
T: end of period

With interest rates declining from 7% to 5% or increasing to 10% the quote of the contract would change from 100 to 101.9 or 97.3 during the time period from $t = t_o$ to $t = T$ ($T - t_o = 1$ year). On T we can assume that the holder of the interest rate option would possibly exercise his option and wish to borrow or lend at the termed exercise interest rate of the option. At the end of the borrowing or lending period which is the "tenor" of the option, the holder of the option would then realise the cost savings of borrowing at the lower than market cap interest rate or realise an additional return by lending at the higher than market floor interest rate. The contract is expected to generate sufficient funds to the writer of the option in order for him to meet his obligations at the end of the borrowing or lending period. The time schedule of interest rate options is shown in fig. 3.

Fig. 3. Time schedule of interest rate option.



If we consider a floor struck at 7% the option is "in the money" when interest rates decline to 5%. The difference of 200 BP would become effective at the end of the period during which the owner of the floor would have invested at 7% interest rate rather than at the lower market interest rate of 5%. If we look at the contract (fig. 2) the change in quotes by 1.9 points from 100 to 101.9 is found to be equal to the present value of two points at the prevailing market interest rate of 5% over a period of one year. The value of the contract of 1.9 points as quoted at time T is then sufficient to hedge the floor since proceeds need only to be invested at the market interest rate of 5% in order to yield two points or 200 BP at the end of the lending period. In this picture the floor is a call on the contract defined in Equation 1 with exercise price equal to 100. A hedge on the floor would involve a long position in the contract.

Similarly, a cap struck at 7% is "in the money" if interest rates increase to 10%. In an analogy to the above, we expect that a short position in the contract can be used as a hedging tool. If interest rates increase to 10%, a short position in the contract results in a gain of 2.7 points (100 − 97.3, see fig. 2) which is equal to the present value of three points or 300 BP at the market interest rate of 10%. The cap corresponds to a "put" option written on the contract with the exercise price at 100.

Fig. 4. The evolution of the neutral hedging strategy for two interest rate outcomes

$$\Delta\,[F\,(S_o \times e^{-d\,(T-t_o)}, E) - F\,(S_o, E)] + e^{r\,(T-t_o)} \times B$$

$$\Delta\,[F\,(S_o, E) - F\,(S_o, E)] + B$$

$$\Delta\,[F\,(S_o \times e^{u\,(T-t_o)}, E) - F\,(S_o, E)] + e^{r\,(T-t_o)} \times B$$

$$\text{with}\quad F\,(S, E) = \frac{1+E}{1+S} \times 100 \qquad \text{(see Eq. 1)}$$

S_o: = Market interest rate initially on time $t = t_o$
E : = Exercise interest rate of option
d : = Decline of interest rates in percentage terms
u : = Increase of interest rates in percentage terms
B : = Denomination of a deposit
Δ : = Hedging ratio
r : = Riskless deposit rate
e : = Exponential function

The neutral hedging strategy that the writer of the interest rate option would pursue is shown in fig. 4 (compare with fig. 2 and fig. 3).

The writer of the option assumes Δ positions[6] in the contract, long for a writer of a floor and short for a writer of a cap. As on entering a futures po-

[6] Delta Hedging is a method option writers use to hedge risk exposure of written options by purchase or sale of the underlying asset in proportion to the delta. For example, a call option writer who has sold an option with a delta of 0.5 may engage in delta hedging by purchasing an amount of the underlying instrument equal to one-half of the amount of the underlying that must be delivered upon exercise. A delta-

sition, there is no cash flow involved at this time. The proceeds of the sale from the option are fully invested in a deposit of denomination B. Initially, on time $t = t_o$, the value of the option C is therefore:

$$C = B \tag{2}$$

V. Floor Option as an Example

For simplicity, we assume that the option expires on T such that we can determine the final values of C. For a floor, which we identified as a call option, the following equations are obtained.

(3) Interest rates decline from S_o to $S_o \times e^{-d(T-t_o)}$:

$$C_u = \max\left[F\left(S_o \times e^{-d(T-t_o)}, E\right) - 100{,}0\right]$$

(4) Interest rates increase from S_o to $S_o \times e^{u(T-t_o)}$:

$$C_d = \max\left[F\left(S_o \times e^{u(T-t_o)}, E\right) - 100{,}0\right]$$

Referring to fig. 2, $C_u = 1.9$ and $C_d = 0$ in the example above. Combining equations three and four with the equations on the right hand side of fig. 3 representing the value of the equivalent portfolio under either interest rate scenario, permits us to determine the hedge ratio Δ[7] and the deposit denomination B. In particular we obtain:

$$\Delta = \frac{C_u - C_d}{F\left(S_o \times e^{-d(T-t_o)}, E\right) - F\left(S_o \times e^{u(T-t_o)}, E\right)} \tag{5}$$

$$B = C \qquad \text{(equ. 2)}$$

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neutral position is established when the writer strictly delta-hedges so as to leave the combined financial position in options and underlying instruments unaffected by small changes in the price of the underlying. Gamma (Γ) gives the sensitivity of an option's delta to small unit changes in the price of underlying. Some options traders attempt to construct "gamma-neutral" positions in options (long and short) such that the delta of the overall position remains unchanged for small changes in the price of the underlying instrument. Using this method writers can produce a fairly constant delta and avoid the transactions costs involved in purchasing and selling the underlying as its price changes. Bank for International Settlements, Recent Innovations in International Banking, April 1986.

[7] Delta (Δ) is the change in an option's price divided by the change in the price of the underlying instrument. An option whose price changes by \$ 1 for every \$ 2 change in the price of the underlying has a delta of 0.5. At-the-money-options have deltas near 0.5. The delta rises toward 1.0 for options that are deep-in-the-money, and approaches 0 for deep-out-of-the-money-options. Bank for International Settlements, Recent Innovations in International Banking, April 1986.

(6)
$$B = \left(C_u \times \frac{1 - dn}{up - dn} + C_d \times \frac{up - 1}{up - dn} \right) \times e^{-r(T - t_o)}$$

with
$$up = \frac{1 + S_o}{1 + S_o \times e^{-d(T - t_o)}}$$

$$dn = \frac{1 + S_o}{1 + S_o \times e^{u(T - t_o)}}$$

The deposit denomination B is equal to the call value as stated in equation 2. Using the numbers in fig. 1 and fig. 2 for a numerical example and assuming a deposit rate of 7%, the floor value and Δ would be calculated as follows:

(7)
$$C = \left(1.9 \times \frac{1 - 0.973}{1.019 - 0.973} + 0 \right) \times \frac{1}{1.07}$$

$$C = 1.048$$

and
$$\Delta = 0.41$$

Equation 6 is familiar to those who have used the *Cox-Rubinstein* method for pricing options on futures contracts. The *Fischer Black* option pricing formula can be obtained from this approach if the upward and downward movements "*up*" and "*dn*" are equal in absolute values. This is not the case here for options on interest rates. The Fischer Black option pricing formula would give only approximate values for the option price.

VI. The Equivalent Portfolio for a Written Floor

The performance of the neutral hedging strategy is tested below with the numbers of fig. 1 and fig. 2. We assume that the writer of a floor has deposited the proceeds of the sale of the option at 7% and entered $\Delta = 0.41$ long positions in the contract. The option expires on T when interest rates are either at 5% (floor "in the money") or at 10% (floor has no value).

(A) Interest Rates at 5%:

Proceeds of deposit (1.048 at 7%)	1.121
gain on 0.41 contract positions	0.779
	1.900

Deposit at 5% over 1 period until delivery (1 year assumed)	2.00	=	200 BP
200 BP due on exercised floor			(200 BP)
Balance			0

(B) Interest rates at 10%:

Proceeds of deposit (1.048 at 7%)	1.121
loss on 0.41 contract positions	(1.121)
(no liability due on written floor)	0

Results are similar for a hedged cap. If financial futures quoted on a discount basis are used for hedging (*T*-Bill Futures, Eurodollar Futures) the Δ hedging ratio derived from this model needs a correction. The reason for this is that discount futures contracts for a given move in interest rates do not provide sufficient funds to meet obligations for written options. This can easily be checked using the presented approach. Hedges with discount futures should be established with $(1 + E) \times \Delta$ positions with E as the exercise interest rate of the option.

VII. Inputs and Applications

A) Interest Rate Volatility

The richness or cheapness of options in general and caps or floors in particular is captured through implied volatility. This is the volatility measure which must be used as input into an option pricing model in order to obtain the current market price of the option. Implied volatility reflects market sentiment. The market is bullish when options are expensive thus implied volatility is high in comparison with historical volatility. The reverse holds true for bearish market sentiment. Historical volatility can only serve as a benchmark measure. By comparing implied with historical volatility one may however derive a level of confidence with regard to how well the hedge on the option is likely to perform.

When the market is using volatility as the only measure to describe risk it effectively accepts the assumption of a log-normal density function for security prices or contract quotes. In table 1 we have listed historical volatilities in various currencies calculated for a time horizon of 5 years. It can be seen that 1 month interest rates are much more volatile than 3 month and 6 month interest rates. Interest rate volatilities in Swiss Franc are very high, nearly twice as high as volatilities in other currencies. In this currency interest rates have ranged from 1/8% to 11% over the 5 year period in consideration. For illustrative purposes the calculation of interest rate volatility is shown in appendix A.

Table 1

Interest Rate Volatilities in various currencies over the last five years
(1. 5. 81 - 1. 5. 86)

Currency	1 month	3 month	6 month
USD	24.51%	19.99%	20.14%
B Pound	41.11%	21.24%	22.92%
SF	72.94%	41.34%	16.61%
DM	41.34%	19.23%	16.91%

B) Forward Interest Rates

The method used to calculate forward interest rates is explained in Appendix B. The tenor of the interest rate option determines the maturity of forward rates. For a three month tenor and for cash interest rates prevailing on the 12th August 1986 the results of forward rate calculations are shown in Fig. 5 and Fig. 6. The pricing examples discussed later are based on these yield curves.

From the upward sloping term structure at that time we expect forward rates to be higher than cash interest rates. The comparison of Fig. 5 with Fig. 6 confirms this. The forward rate curve also shows several peaks. This structure occurs whenever the cash yield curve is changing its slope between two maturities. A numerical smoothing procedure can be used, of course, to get rid of this structure. We do not favour this approach, we would prefer to maintain the genuine environment in which the interest rate option is written. Closer spaced maturities of cash interest rates achieve this in a better way.

The evaluation of an interest rate option with a life of several years requires the consideration of a series of individual European options each written on the respective forward interest rate. The total value of the interest rate option is then equal to the sum of the individual option values discounted back to present value. The discount factors we have used are the cash interest rates of maturities equal to the date the respective option

Fig. 5. **Cash Interest Rates**
The cash Yield Curve in US-Dollar as of 12th August 1986

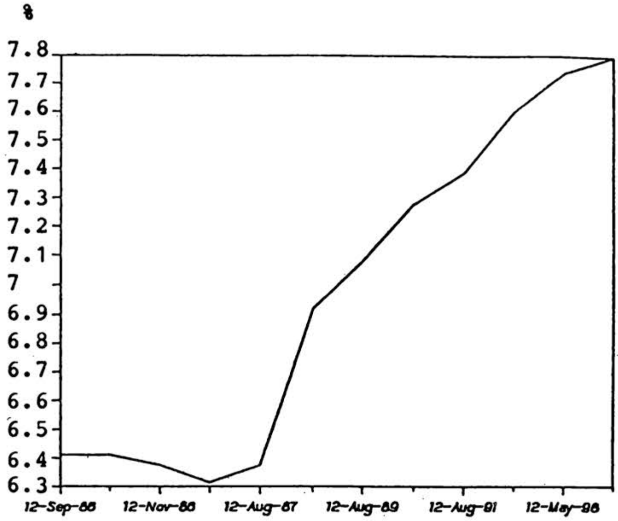
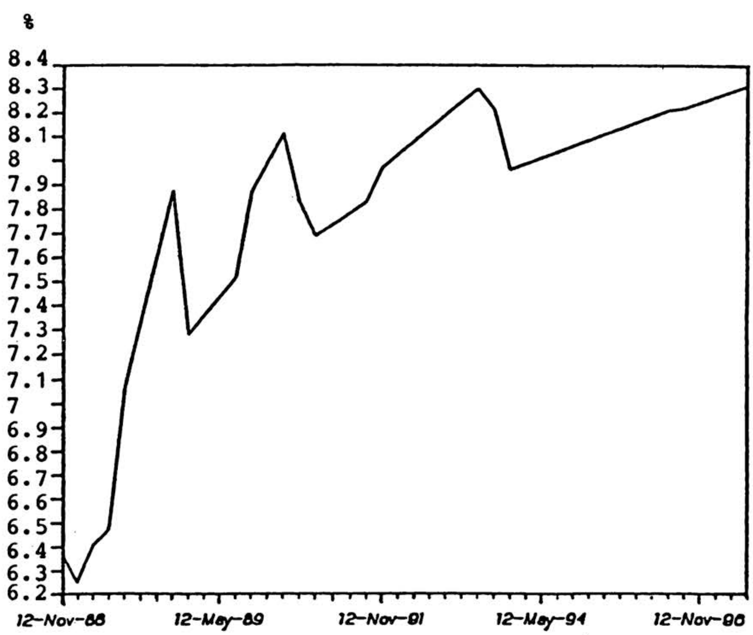


Fig. 6. **Forward Interest Rates**
The Forward Yield Curve derived from data plotted in Fig. 5 for a 3 month tenor
Date: 12th August 1986



becomes effective. We thus obtain a spectrum of option prices and Δ factors extending over the life of the interest rate option.

To be precise one should distinguish between bid and offer interest rates. Floors in particular are written on forward bid and caps on forward offer interest rates while the discount factor is calculated from offer rates. There are some difficulties in calculating forward bid and offer rates as mentioned in the Appendix. The applications presented in the following are based on mean rates.

Table 2

Comparison Market Prices of US-Dollar Caps versus Theoretical Prices.
The underlying cash- and forward yield curves as of 12th August 1986 are shown in Fig. 5 and Fig. 6

Maturity of Cap	3 years	3 years	3 years
Tenor	3 month (LIBOR)	3 month (LIBOR)	3 month (LIBOR)
Exercise interest Rate	7.5%	8%	8.5%
Trading Price .Bid/Offer (upfront BP)	165 BP/ 180 BP	120 BP/ 140 BP	98 BP/ 110 BP
Theoretical Cap Prices with			
20% volatility	150 BP	121 BP	92 BP
22.5% volatility	178 BP	141 BP	111 BP
25% volatility	199 BP	162 BP	131 BP

Maturity of Cap	4 years	5 years	5 years	10 years
Tenor	3 month (LIBOR)	3 month (LIBOR)	3 month (LIBOR)	3 month (LIBOR)
Exercise interest Rate	10%	8%	9%	9%
Trading Price .Bid/Offer .upfront BP	75 BP/ 95 BP	370 BP/ 385 BP	250 BP/ 275 BP	640 BP/ 700 BP
Theoretical Cap Prices with				
20% volatility	77 BP	289 BP	193 BP	586 BP
22.5% volatility	102 BP	331 BP	234 BP	685 BP
25% volatility	129 BP	373 BP	275 BP	784 BP

C) Trading Prices of US-Dollar Caps Compared with Theoretical Prices

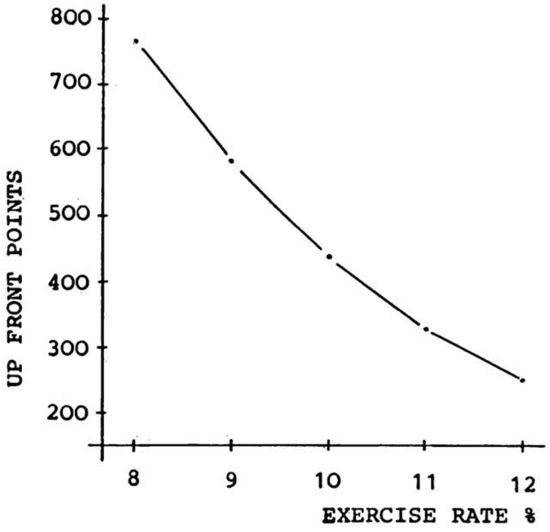
Table 2 contains a list of trading prices of caps on 3 month LIBOR in US-Dollar. Maturities extend from 3 years to 10 years. The narrow Bid/Offer spread at which these instruments are trading is surprising. Debt warrants, for instance, are trading at Bid/Offer spreads of at least 50 BP. We have compared these trading prices with theoretical prices generated by the presented model. We have performed the calculation with various volatilities in order to demonstrate the sensitivity of cap prices on this input parameter. The general picture is that prices on the offer side can be well reproduced with input volatilities at 22.5%. Keeping in mind that historical volatility of US-Dollar 3 month LIBOR is approximatively 20 pct it becomes obvious that traders require a "haircut" in the order of 20 BP, which corresponds roughly to plus 2.5% interest rate volatility. The model helps to identify expensive interest rate options as the example of the 5 year 8% cap shows. This cap appears to be overpriced by approximatively 40 BP. Even with an input volatility of 25 pct which gives us a theoretical price of 373 BP we remain below the trading price of 385 BP. From the global picture obtained from the cap prices in the table, we would expect a trading price in the order of 330 BP. The 10 year cap at 9% LIBOR seems to be quite cheap in view of its long life.

D) Dependence on Exercise Rate

We have calculated the value of caps for different exercise rates. The underlying term structure is shown in fig. 5 and fig. 6. The results of these calculations are plotted in fig. 7.

Cap prices decline with increasing exercise interest rates. The relationship is however not linear. We expect a linear relationship when the underlying term structure is flat. In this case, each period of the interest rate option has equal weight in the total value of the option. More distant periods of the option contribute less due to a smaller discount factor. The concave curvature of the graph in fig. 7 points to an upward sloping term structure. In an environment of this kind mainly the more distant periods of the option contribute to the cap price when the exercise rate is high. This contribution becomes more important the higher the exercise rate of the cap which is indicated by the concave curvature.

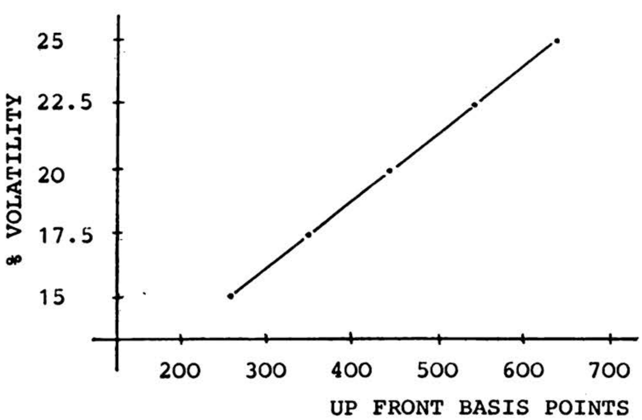
Fig. 7. The value of a Cap as a function of exercise rate.
Volatility = 20%, Maturity = 10 years, 3 month tenor



E) Dependence on Volatility

Cap prices are linearly dependent on volatility, as can be seen in fig. 8. Option prices on stocks and futures calculated with the *Black-Scholes* or *Fischer Black* model show a similar relationship as a function of volatility.

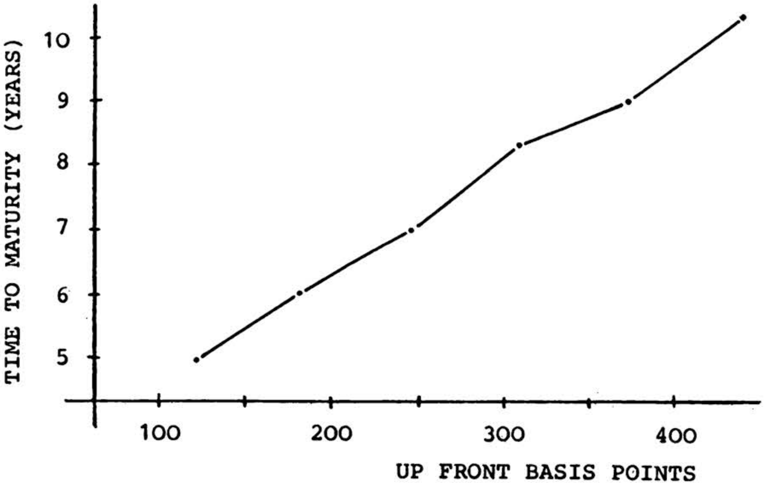
Fig. 8. The value of a Cap as a function of volatility.
Exercise rate = 10%, Maturity = 10 years, 3 month tenor



F) *Dependence on Time to Maturity*

The relation of cap prices to time to maturity is shown in fig. 9. A longer life increases the value of an option as the positive slope of the graph in fig. 9 illustrates. The relationship is however not fully linear as we expect from other option pricing models such as *Black-Scholes* or *Fischer Black*. When the maturity of an interest rate option is increased the shape of the yield curve determines essentially to what extend the distant periods of the option contribute. In a negatively sloped yield curve environment, which would lead us to predict interest rates in decline, the contribution to cap prices from distant periods may even vanish. The reverse holds true for floor options.

Fig. 9. The value of a Cap as a function of time to maturity.
Volatility = 20%, Exercise rate = 10%, 3 month tenor



Conclusion

We have benefited from the flexibility of the *Cox-Rubinstein* option pricing concept to derive a method for pricing interest rate options. We have defined the instrument underlying an interest rate option which allows us to single out the strategies to be followed for hedging the interest rate option. In the context of the equivalent portfolio the similarity of swap agreements and interest options becomes obvious. This has been the basis of the model presented in this paper. Cap and floor prices are rapidly calculated with this procedure even though we have not obtained an explicit formula for options prices. No other input is required than interest rate volatility and the current yield curve. We have tried to reduce subjective input to the minimum. We have found that market prices are well in line with prices calculated from this model. It has been possible to point out slightly overvalued caps on the market. This information may become valuable once the question of how to hedge an interest rate option over its full life is considered more deeply.

Appendix A

Calculation of Interest Rate Volatility

From a time series of interest rates the estimate of mean and variance of proportional changes is calculated as follows:

$$\text{Mean} = \frac{1}{n} \sum_{k=1}^{n} \ln (P_k)$$

$$\text{Variance} = \frac{1}{(n-1)} \sum_{k=1}^{n} (\ln^2 (P_k) - \text{mean}^2)$$

$$\text{With } P_k = \frac{S_{j+1}}{S_j} \qquad j = 1, 2, \ldots, N$$

S_{j+1}, S_j = subsequent observations of interest rates

N = total number of interest rates in time series

n $= N - 1$

The mean becomes zero the shorter the time between two observations. The reason is that the true mean and variance which are approximated here through calculations based on historical data, are both proportional to the time elapsed between two observations. On a weekly or even daily basis the proportional changes and their mean become small. The square of the mean, which enters into the calculation of the variance, is even smaller. From a sample of closely spaced observations of interest rates we obtain an estimate of the variance of proportional changes which is widely independent of its mean. The calculation of the mean can be then omitted.

The volatility of interest rates is calculated taking the square root of the variance. This is the standard deviation of proportional changes.

$$\text{Standard Deviation} = \text{std} = \sqrt{\text{var} (\ln (P_k))}$$

Volatility of interest rates is the annualized figure of the standard deviation. Using daily observations, as an example, one has to multiply the variance with 252, number of business days per year.

Volatility determines how widely apart interest rates are branching in a *Cox-Rubinstein* option pricing model.

Appendix B

Calculations of Forward Rates

The calculation of forward rates is essentially a break even consideration. For example, the forward bid rate on a delivery date of the option is the interest rate at which short term funding is balanced by future short term lending such that initial long term lending is replicated.

We need to consider a time period T of long duration and a time period t of short duration. We assume that we can obtain interest rates i_T and i_t from the cash yield curve with terms T and t respectively. We want to calculate the forward interest rate i_f of tenor $(T - t)$ which we expect to be the short term interest rate t-years from now on. This calculation is easier with continuously compounding (zero coupon) rates which we obtain as follows:

$$i'_T = \ln(1 + i_T)$$

$$i'_t = \ln(1 + i_t)$$

with $i'_{T,t}$ = annual continuously compounding interest rates

$i_{T,t}$ = annual interest rates, annually compounding

ln = natural logarithm

The forward interest rate on continuously compounding basis is obtained from:

$$i'_f = \frac{1}{(T - t)} * (i'_T \cdot T - i'_t \cdot t)$$

with i'_f = annual continuously compounding forward interest rate

This is recalculated into an annual compounding interest rate by:

$$i_f = 100 * (e^{i'_f} - 1)$$

with i_f = annual compounding forward interest rate

2*

The numerical procedure of pricing interest rate options in the *Cox-Rubinstein* frame work is generally facilitated by making use of continuously compounding interest rates.

The forward bid rate is calculated with i_T equal to the bid and i_t equal to the offer rate on the cash yield curve. Large spreads between cash bid and offer rates distort the calculation of the respective forward rates however. In general, we have used mean rates for calculations of cap and floor prices.

Zusammenfassung

Notierung von „Caps“ and „Floors“ Ein vereinfachtes Verfahren

Das Optionspreismodell von *Cox-Rubinstein* diente der Erarbeitung einer Methode für die Notierung von „Caps“ und „Floors“. Dabei handelt es sich um Zinsoptionen, die den Zins-Swaps eng verwandt sind. Dieses Charakteristikum wird als Grundlage für die Definition eines neuen Instruments herangezogen, das der Zinsoption zugrunde liegt. Ausgehend von der auf diesem Instrument beruhenden neutralen Kurssicherungsstrategie kann das Modell entwickelt werden. Es erweist sich, daß der Tageskurs für „Caps“ und „Floors“ erhalten bzw. als zu hoch oder zu niedrig im Markt bewertet wird. Das Modell erfüllt somit seinen Zweck und weist auch auf die zu empfehlenden Kurssicherungsstrategien hin.

Summary

Pricing of Caps and Floors A Simplified Approach

The *Cox-Rubinstein* option pricing model has been used to derive a method for pricing caps and floors. Caps and floors are interest rate options closely related to interest rate swaps. This feature has been used as the basis for defining a new instrument underlying the interest rate option. From the neutral hedging strategy based on this instrument, the model can be developed. It turns out that market prices of caps or floors can be reproduced or distinguished as too high or too low. The model thus fulfills its purpose and also indicates which hedging strategies to pursue.

Résumé

Fixation de « caps » et « floors »

Le modèle de fixation de prime de *Cox-Rubinstein* a été utilisé pour en dériver une méthode en vue de calculer les « caps » et « floors ». Les caps et floors sont des primes de taux d'intérêt, fort semblables à des swaps de taux d'intérêt. Cette caractéristique

a été employée comme base pour définir un nouvel instrument sur lequel est fondé l'option de taux d'intérêt. A partir de la stratégie de couverture (hedging) neutre, basée sur cet instrument, on a développé le modèle. Celui-ci montre que les prix de marché des caps ou floors peuvent être reproduits ou caractérisés trop élevés ou trop bas. Le modèle remplit donc son objectif et indique aussi quelles stratégies de couverture poursuivre.