A Generalized Production Function and its Special Cases

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I. Introduction

In a pioneering effort to determine to what extent capital and labor are substitutable for each other in production, *Arrow, Chenery, Minhas* and *Solow* derived the constant elasticity of substitution production function.[1] This function, usually referred to as CES production function, enriched economic theory considerably. Its properties were subsequently explored in a number of articles and some extensions to it have been offered.[2]

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[1] K. J. *Arrow*, H. B. *Chenery*, B. S. *Minhas* and R. M. *Solow*, "Capital-Labor Substitution and Economic Efficiency", The Review of Economics and Statistics, August 1961, Number 3, p. 225 to 250. The footnote 7 in this article states that Trevor *Swan* independently deduced the constant-elasticity-of-substitution property and that *Solow* used the function itself as an illustration in a Quarterly Journal of Economics article in 1956.

[2] See for instance: *Murray, Brown* and John S. *deCani*, "Technological Change and the Distribution of Income", International Economic Review, Vol. 4, No. 3, September 1963, p. 289 - 309. Giora *Hanoch*, "CRESH Production Functions", Econometrica, Vol. 39, No. 5, Sept. 1971. *McFadden*, D. "Constant Elasticity of Substitution Production Functions", Review of Economic Studies, 1963, p. 73 - 83. Jacob *Paroush*, "A Note on the CES Production Function", Econometrica, Vol. 32, No. 1 - 2, January - April 1964, p. 213. W. M. *Gorman*, "Production Functions in which the Elasticities of Substitution stand in Fixed Proportions to each other", Review of Economic Studies 1965, p. 217 - 224. J. K. *Whitacker*, "A Note on the CES Production Function", Review of Economic Studies, 1964, p. 166 - 167. G. C. *Harcourt*, "Biases in Empirical Estimates of the Elasticities of Substitution of the C. E. S. Production Functions", Review of Economic Studies, 1966, p. 227 - 233. H. *Uzawa*, "Production Functions with Constant Elasticities of Substitution", Review of Economic Studies, 1962, p. 291 - 299.. V. *Mukerji*, "A Generalized SMAC Function with Constant Ratios of Elasticities of Substitution", Review of Economic Studies, 1963, p. 233 - 236 (where an extension of the CES function with more than two factors of production is attempted). *Murray, Brown*, "A Measure of the Change in Relative Exploitation of Capital and Labor", Review of Economics and Statistics, Vol. 48, 1966, p. 182 - 192. P. J. *Dhrymes*, "Some Extensions and Tests for the CES Class of Production Functions", Review of Economics and Statistics, Nov. 1965, p. 357 - 366.

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In this paper an effort is made to show that the CES function is a special case of a still wider class of first degree homogeneous production functions. There exist more general production functions which can allow for departures from competitive marginal cost pricing and adjustments (or better difficulties in adjustments) of the capital-labor ratio in response to changes in factor prices. Before these functions are derived it is helpful to describe shortly how the CES production function, which we shall use as a convenient starting point, was originally found.

II. The Derivation of the CES Production Function

The derivation of the CES production function involved two distinct steps. The first one was an empirical study of the relationship between labor productivity and wages. It was observed that the value added per unit of labor used in a given industry varies across countries with the wage rate. *Arrow, Chenery, Minhas* and *Solow* reported that a regression of labor productivity on the wage rate, in a study covering 24 manufacturing industries in 19 countries, showed a highly significant correlation in all industries and also a considerable variation in the regression coefficients. The hypothesis was that wages change labor productivity, the causal flow being from the former to the latter. A positive correlation meant higher wages had an incentive effect causing more and better work to be forthcoming. In this first step of the study no assumptions were made as to how wages were determined. The empirical study was a general one. It could have been applied to market as well as non-market economies.

Estimated, and used for further analysis, was the equation

$$\log \frac{V}{L} = \log a + b \operatorname{Log} W + \varepsilon \tag{1}$$

where the symbols have the following meaning:

V = value added in current prices, in dollars
L = labor-input in man-years
W = money wage rate per man-year, in dollars
a, b = coefficients to be estimated
ε = random disturbance term

The second step in the derivation of the CES function is based on the assumption that the observed configuration of labor productivity and wages is the outcome of a competitive neoclassical market process. It is assumed that there is an underlying production function of the general form:

$$V = F(K, L) \tag{2}$$

where L denotes labor, and K capital.

The function remains at first unspecified, except for the requirement that it shall be homogeneous of degree one.

Setting $\frac{V}{L} = y, \frac{K}{L} = x$ and denoting wages with w, it can be shown that $y = F(x)$ and further:

$$w = F(x) - xF'(x) \qquad \text{or} \tag{3}$$

$$w = y - x\frac{dy}{dx} \tag{4}$$

Substituting equation (4) into (1) and taking antilogarithms on both sides yields the basic differential equation

$$y^{\frac{1}{b}} = a^{\frac{1}{b}}\left(y - x\frac{dy}{dx}\right) \tag{5}$$

Integration of (5) leads to the CES production function:[3]

$$V = (\beta K^{-p} + a^{*} L^{-p})^{-\frac{1}{p}} \tag{6}$$

where: β is a constant of integration

$$a^{*} = a^{-\frac{1}{b}}$$

$$p = \frac{1}{b} - 1$$

b is the elasticity of substitution.

III. The Generalized Production Function

The functional form of equation (1) is based on the hypothesis that labor productivity (value added per unit of labor input) is determined

[3] Equation (6) is a form of the CES production function given in *Arrow, Chenery, Minhas, Solow*, op. cit., formula (11) p. 230. We replaced a in their notation by our a^{*}.

by the wage rate. In the ensuing analysis the discoverers of the CES function then assumed that the observed wage rates are competitive ones.

We propose to change above approach in two respects. The first one concerns the wage rate. We retain the hypothesis that labor productivity depends on the wage rate, yet we shall not insist that it is competitively determined. Departures from the competitive wage rate shall be possible because the factors of production (labor and capital) may possess some degree of market power.

We write equation (4) therefore as:

$$w^* = y - \tau\, x \frac{dy}{dx} \tag{7}$$

where: w^* stands for the non-competitive wage and the τ coefficient indicates a departure from the perfect competition marginal productivity of capital, the real rental of capital.

Three cases can be distinguished:

(a) if $\tau = 1$ $w^* = w$
(b) if $\tau > 1$ $w^* < w$
(c) if $\tau < 1$ $w^* > w$

Case (a) is the standard neoclassical case of perfect competition. Case (b) indicates that capital succeeds in obtaining a remuneration in excess of its marginal product. In case (c) wages are higher than the marginal product of labor.

The second change concerns the adjustment of the capital-labor ratio. The *Arrow, Chenery, Minhas, Solow* analysis is in nature a neo-classical long-run one. It assumes that the capital-labor ratio has adjusted in line with factor prices determined by long-run marginal productivity. An equilibrium is reached. The ultimate long-run capital-labor ratio may however not be equal to observed short-run ones. A discrepancy may arise because capital in place often cannot be adjusted quickly and (or) rapid changes in labor inputs may not be feasible. The observed capital-labor ratio shall have a direct effect on productivity. We rewrite therefore equation (1) as

$$\log \frac{V}{L} = \log a + b \log w + c \log \frac{K}{L} + \eta \tag{8}$$

The symbols have the same meaning as in (1). The additional variable K denotes capital. The additional parameter to be estimated is c and η is the new random disturbance. If c should turn out to be significant — and some preliminary studies indicate that it will — the capital-labor ratio should be kept as an explanatory variable. In terms of our equation (6) a discrepancy between the short-run and long-run capital-labor ratio implies that in the short-run $c \neq 0$. If adjustments in the capital-labor ratio take place over time, c should decrease. In the long-run, when all adjustment are achieved, c is equal to zero.

Substituting (7) into (8) we obtain

$$\log y = \log a + b \log \left(y - \tau x \frac{dy}{dx}\right) + c \log x \tag{9}$$

where $y = \frac{V}{L}$ and $x = \frac{K}{L}$

as before.

Taking antilogarithms on both sides of equation (9) we get:

$$y = a x^{c} \left(y - \tau x \frac{dy}{dx}\right)^{b} \tag{10}$$

This differential equation is basic for the following analysis.

IV. First Degree, Homogeneous Production Functions

Through integration of (10) a general, short-run production function is obtained. The wage rate shall be non-competitive ($\tau \neq 1$) and the capital-labor ratio shall not yet have adjusted to this wage rate ($c \neq 0$). In this situation the production function is:

$$V_1 = \left[\frac{\frac{p}{\frac{1}{b}}}{a \tau \left(\frac{1}{b} - 1\right)} K^{-\frac{c}{b}} L^{\frac{c}{b} - p} + \beta K^{-\frac{p}{\tau}} L^{\frac{p}{\tau} - p} \right]^{-\frac{1}{p}} \tag{11}$$

where: $p = \frac{1}{b} - 1$

β is a constant of integration

The first special case of (10) arises if wages are determined competitively. In this case $\tau = 1$ and (10) reduces to

(12) $$V_2 = \left[\frac{p}{a^{\frac{1}{b}} \left(p - \frac{c}{b}\right)} K^{-\frac{c}{b}} L^{\frac{c}{b} - p} + \beta K^{-p} \right]^{-p}$$

The problem of the capital-labor ratio adjustment remains and $c \neq o$.

The second special case arises if $c = o$ but $\tau \neq 1$. In this situation the capital-labor ratio did adjust in line with a non-competitive wage rate. The production function becomes:

(13) $$V_3 = \left[\frac{1}{a^{\frac{1}{b}}} L^{-p} + \beta K^{-\frac{p}{\tau}} L^{\frac{p}{\tau} - p} \right]^{-\frac{1}{p}}$$

The third special case is the conventional *Arrow, Chenery, Solow, Minhas* CES production function. It arises if wages are competitive ($\tau = 1$) and the capital-labor ratio has adjusted in line with the wage rate ($c = 0$). The production function is now:

(14) $$V_4 = \left[\frac{1}{a^{\frac{1}{b}}} L^{-p} + \beta K^{-p} \right]^{-\frac{1}{p}}$$

In all cases so far it was assumed that the elasticity of substitution b falls between the following values:

$$0 < b < \infty$$

because $p = \frac{1}{b} - 1$ this implies

$$\infty > p > -1$$

This elasticity assumption underlies the CES function.

For the sake of completeness three special cases of the CES function may be shortly mentioned because they yield widely used production functions. If $\tau = 1$, $c = 0$, and $b = 1$, ($p = 0$), differential equation (10) simplifies to:

(14a) $$y = a\left(y - x \frac{dy}{dx}\right)$$

Integrating (14a) yields the well-known Cobb-Douglas production function

$$V_5 = \beta\, K^{1-a} L^{a} \tag{15}$$

If $\tau = 1$, $c = 0$, $b = \infty$, ($p = -1$) a perfect substitution production function results. Its form can be easily found by substitution -1 for p in (14). One obtains:

$$V_6 = L + \beta K \tag{16}$$

Finally, if $\tau = 1$, $c = 0$, $b = 0$ ($p = \infty$) the Leontief fixed proportion production function can be obtaind from (14) through a limiting process. Its general form is:

$$V_7 = \gamma \min [K, L] \tag{17}$$

where γ is a constant.

Zusammenfassung

Eine verallgemeinerte Produktionsfunktion und ihre Sonderfälle

Vor etwa sechzehn Jahren leiteten *Arrow, Chenery, Minhas* und *Solow* eine Produktionsfunktion mit konstanter Faktorsubstitutionselastizität ab (CES production function). Die Ableitung erfolgte in zwei Schritten. Zunächst wurde in einer stochastischen Gleichung die Arbeitsproduktivität durch den Lohnsatz erklärt. In dem zweiten Schritt wurde dann eine Verbindung zwischen dieser empirischen Gleichung und der traditionellen neoklassischen Theorie hergestellt: Es wurde angenommen, daß der Lohnsatz das Ergebnis einer vollkommenen Konkurrenz und eine Funktion der capital-labor-ratio ist.

In diesem Aufsatz wurde das obige Ableitungsverfahren modifiziert. Es wurde zunächst angenommen, daß die Arbeitsproduktivität vom Lohnsatz und der capital-labor-ratio bestimmt wird. Ferner wurde angenommen, daß der Lohnsatz unter Bedingungen einer unvollkommenen Konkurrenz zustande kommt. Mit diesen Änderungen erhält man eine viel allgemeinere Klasse von linear homogenen Produktionsfunktionen. Diese schließt als Sonderfälle die „Variable Substitutionselastizitäts-Produktionsfunktion (VES production function) und die CES Funktion ein. Es ist wohlbekannt, daß die letztere wiederum als Sonderfälle (a) die perfekte Substitutionselastizitäts-Produktionsfunktion, (b) die *Cobb-Douglas*-Produktionsfunktion und (c) die *Leontief* (fixed proportion) Produktionsfunktion hat.

Summary

A Generalized Production Function and its Special Cases

Sixteen years ago *Arrow, Chenery, Minhas* and *Solow* derived the Constant Elasticity of Substitution (CES) production function. Its derivation proceeded in two steps. The first one was to estimate a stochastic equation in which labor productivity is a function of the wage rate. The second step involved a linkage of this empirically determined equation with the body of established neoclassical theory: the wage rate was assumed to be determined competitively and a function of the capital-labor ration.

In this paper above procedure for the derivation of the CES function was modified. First, the empirical proposition is that labor productivity is determined by the wage rate and the capital-labor ratio. Second, the wage rate is assumed to be determined in a non-competitive manner.. With these two changes a much wider class of linear homogeneous production functions is obtained. It includes as special cases the Variable Elasticity of Substitution (VES) production function and the CES function. As is wellknown, the latter in turn includes as special cases (a) the perfect elasticity of substitution production function, (b) the *Cobb-Douglas* production function and (c) the *Leontief* fixed proportion production function.

Résumé

Une fonction généralisée de production et ses cas exceptionnels

Il y a quelque seize années, *Arrow, Chenery, Minhas* et *Solow* induisaient une fonction de production avec un facteur constant d'élasticité de substitution (CES production function). Cette dérivation s'effectua en deux étapes. D'abord une équation stochastique expliqua la productivité du travail par le taux de salaire. Ensuite fut établie une relation entre cette équation empirique et la théorie néoclassique traditionnelle: l'on a supposé que le taux de salaire était le fruit d'une concurrence parfaite et une fonction de la relation capital-travail.

Le présent article modifie le procédé d'induction précité. L'on admet que la productivité du travail est déterminée par le taux de salaire et par la relation capital - travail. Et l'on admet également que le taux de salaire s'établit dans les conditions d'une concurrence imparfaite. Ces modifications permettent d'obtenir une classe plus générale de fonctions de production linéaires homogènes. Cette classe inclut comme cas exceptionnels la « fonction de production variable de substitution élastique » (VES production function) et la fonction CES. Il est bien connu que cette dernière a par ailleurs comme cas d'exception (a) la fonction parfaite de production de substitution élastique, (b) la fonction de production Cobb-Douglas et (c) la fonction de production Leontief (fixed proportion).

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