Zeitschrift für Wirtschafts- u. Sozialwissenschaften (ZWS) 119 (1999), S. 45 – 55
Duncker & Humblot, Berlin

On the Stability of the Balanced Growth Path in the Solow Model*

By Bernhard Rauch**

1. Introduction

Solow's (1956) growth model is the fundamental model in neoclassical growth theory. It assumes a constant saving rate s and an exogenously given growth rate n of efficient labor L, and it is well known that capital intensity $k = K/L$ and per capita income $y = Y/L$ converge to their equilibrium values k^* and y^* respectively.

If we consider additional variables, for which we are mainly concerned with the absolute rather than the per capita value – e.g. environmental pollution resulting from production – the exact time paths of K and Y as well as their deviation from the equilibrium paths gain importance. On the balanced growth path, denoted by $K^*(t)$ and $Y^*(t)$, capital and income grow at the constant rate n. Given an initial deviation from the equilibrium, the growth rates of $K(t)$ and $Y(t)$ converge to n. Intuitively one would think that like $k^*(t) - k(t)$ and $y^*(t) - y(t)$ the differences $K^*(t) - K(t)$ and $Y^*(t) - Y(t)$ would tend to zero for increasing t, as shown in figure 1, which is taken from Krelle (1985, p. 129). This view can be derived from figure VI in Solow (1956) (given here as figure 2) and can also be found in some recent textbooks on growth theory.

If, however, the economy is not yet in equilibrium at time $t = 0$, only $[K^*(t) - K(t)]/K(t)$ and $[Y^*(t) - Y(t)]/Y(t)$ tend to zero. (This implies that the length of time $T_K(t)$ needed for $K(t + T_K(t))$ to equal $K^*(t)$ tends to zero for increasing t. The same applies to Y.) The differences $K^*(t) - K(t)$ and $Y^*(t) - Y(t)$ themselves tend to (plus or minus) infinity, as shown in figure 3. Put another way: if two countries are identical except for their initial capital endowments, the difference in their capital endowments and production will grow without bound. This is proved in the paper.

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* Verantwortlicher Herausgeber / editor in charge: B. F.

** I am indebted to Jürgen Heubes and Winfried Vogt. An anonymous referee gave very valuable comments. Any remaining shortcomings are, of course, my responsibility.

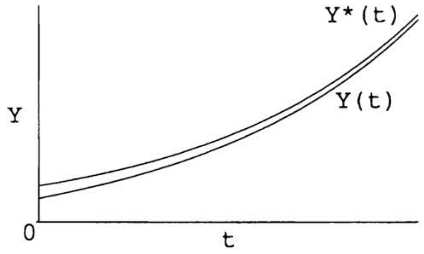

Figure 1

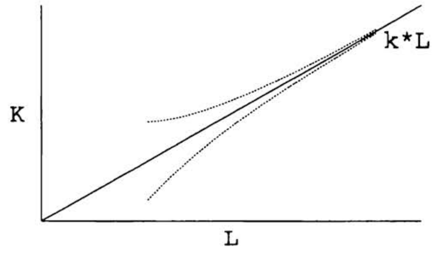

Figure 2

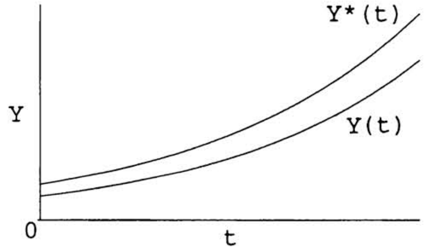

Figure 3

There are three reasons for dealing with this problem: Firstly, the Solow model is the fundamental model in neoclassical growth theory. Hence, it is desirable to completely understand its implications and to correct errors concerning this model contained in the literature (and even in the original article). Secondly, the result has implications for the relationship between economic growth and environmental pollution, which has attracted increasing attention. Thirdly, the result demonstrates that unexpected things can happen in mathematical models where variables tend to infinity as is usual in growth theory.

Section 2 shows the divergence of $K^* - K$ and $Y^* - Y$ and how this divergence is related to the convergence of the per capita values. In section 3 the conditions are derived under which this divergence occurs if the saving decision is endogenous. In section 4 the implications of the result are shown for an extended Solow model with an environmental sector.

2. The Differences $K^* - K$ and $Y^* - Y$

In the Solow model, output is produced using two factors, capital K and labor L. The production function

$$Y = Y(K, L) \tag{1}$$

shows constant returns to scale, so that per capita production y is a function $y(k)$ of capital intensity $k = K/L$. Labor grows at the constant rate n, i.e.

$$L(t) = L_0 e^{nt} \, . \tag{2}$$

There is no capital depreciation and investment is a fixed fraction s of output, so that

$$\dot{K}(t) = sY(t) \, . \tag{3}$$

If the production function satisfies the Inada conditions, there is a unique balanced growth path

$$K^*(t) = K_0^* e^{nt}, \qquad Y(t) = Y_0^* e^{nt} \tag{4}$$

where K and Y grow at rate n, i.e. k and y are constant at their equilibrium values k^* and y^*. k^* and y^* are determined by $(n/s)k^* = y(k^*) = y^*$ and $K_0^* : = k^* L_0, Y_0^* : = y^* L_0$. The balanced growth path is stable in the sense that if $K_0 \neq K_0^*$, $k(t)$ and $y(t)$ converge to their equilibrium values as t tends to infinity, i. e. the deviations $k^* - k(t)$ and $y^* - y(t)$ tend to zero, and the growth rates $w_{K(t)}$ and $w_{Y(t)}$ of $K(t)$ and $Y(t)$ tend to n.

This paper is concerned with the differences $K^*(t) - K(t)$ and $Y^*(t) - Y(t)$. It is easy to see that these differences do not tend to zero given a deviation from the equilibrium. If, for instance, $K(t) < K^*(t)$, then $Y(t) < Y^*(t)$ and consequently $\dot{K}(t) = sY(t) < sY^*(t) = \dot{K}^*(t)$. Production and therefore investments are smaller than on the balanced growth path because of the small capital stock. (Nevertheless $\dot{K}(t)/K(t) > \dot{K}^*(t)/K^*(t) = n$.) Therefore the difference $K^* - K$ is growing. Because the marginal product of capital converges to $y'(k^*) > 0$ monotonously, $Y^* - Y$ can not converge to zero. The following proposition states the divergence of the differences.

P r o p o s i t i o n 1

Set $\alpha^* := y'(k^*)k^*/y(k^*)$. If $K_0 \neq K_0^*$, then

(5) $$\lim_{t\to\infty} w_{K^*(t)-K(t)} = \lim_{t\to\infty} w_{Y^*(t)-Y(t)} = \alpha^* n \,.$$

In particular, the differences $K^*(t) - K(t)$ and $Y^*(t) - Y(t)$ diverge to plus or minus infinity.

P r o o f :

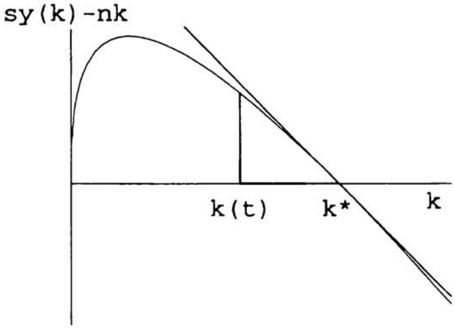

Figure 4

Figure 4 shows $\dot{k} = sy(k) - nk$ as a function of k and the tangent to this function in k^*. It is clear from figure 4 and well known from the literature, e.g. Klump / Maußner (1996), that because of $\lim_{t\to\infty} k(t) = k^*$

(6) $$\lim_{t\to\infty} \frac{\dot{k}(t)}{k^* - k(t)} = -sy'(k^*) + n = (1-\alpha^*)n \,.$$

Because of $\dot{K}^*(t) = nK^*(t)$ and $\dot{k}(t) = [\dot{K}(t) - nK(t)]/L(t)$ we have

(7) $$\frac{\dot{K}^*(t) - \dot{K}(t)}{K^*(t) - K(t)} = n - \frac{\dot{K}(t) - nK(t)}{K^*(t) - K(t)} = n - \frac{\dot{k}(t)}{k^* - k(t)} \,.$$

Therefore (6) implies

(8) $$\lim_{t\to\infty} w_{K^*(t)-K(t)} = \alpha^* n \,.$$

For all $t, \dot{y}(t) = y'[k(t)]\dot{k}(t)$, hence $\lim\limits_{t\to\infty} \frac{\dot{y}(t)}{\dot{k}(t)} = y'(k^*) = \lim\limits_{t\to\infty} \frac{y^*-y(t)}{k^*-k(t)}$. Therefore,

(9) $$\lim_{t\to\infty} \frac{\dot{y}(t)}{y^* - y(t)} = \lim_{t\to\infty} \frac{\dot{k}(t)}{k^* - k(t)}$$

and (7) as well as (8) hold with capital replaced by income. q.e.d.

If we set $g(t) := k(t) - k^*$ and $h(t) := y(t) - y^*$, we can write $K(t) = [k^* + g(t)]L(t)$ and $Y(t) = [y^* + h(t)]L(t)$. It is clear that $g(t)$ and $h(t)$ tend to zero as t tends to infinity. The proposition says that for $K_0 \neq K_0^*$ $g(t)L(t)$ and $h(t)L(t)$ tend to (plus or minus) infinity at a rate approaching $\alpha^* n$. Because the time path of labor is exogenously given, the deviation of capital from the equilibrium value affects production only with a factor approaching $\alpha^* < 1$, so that in spite of the divergence of $K^* - K$ and $Y^* - Y$ capital intensity and income per capita tend to their respective equilibrium values.

The proof of the proposition uses the fact that the convergence rate for k and y according to equations (6) and (9) is equal to $(1 - \alpha^*)n$. The long run growth rate of $K(t)$ is n. The gap between $k(t)$ and $k*$ closes at the rate $(1 - \alpha^*)n$, so the gap between $K(t)$ and $K^*(t)$ opens at the rate $\alpha^* n$. The same applies to $Y(t)$. The proposition, however, can be proved without using equation (6). Thus the proposition provides an alternative for calculating the convergence rate for $k(t)$ and $y(t)$. Therefore I give an alternative proof, which for the sake of simplicity is restricted to the case of the Cobb-Douglas production function $Y = K^\alpha L^{1-\alpha}$, which allows an explicit calculation of the growth path.

For the Cobb-Douglas production function[1] investment at time t is equal to $\dot{K}(t) = sK(t)^\alpha L_0^{1-\alpha} e^{(1-\alpha)nt}$. As shown in Solow (1956, p. 76) the resulting stock of capital is

(10) $$K(t) = \left(K_0^{1-\alpha} - \frac{s}{n} L_0^{1-\alpha} + \frac{s}{n} L_0^{1-\alpha} e^{(1-\alpha)nt}\right)^{\frac{1}{1-\alpha}}.$$

With $A := (K_0/K_0^*)^{1-\alpha} - 1$, (10) yields

(11) $$K(t) = K_0^* e^{nt}(1 + \mathrm{A}e^{-(1-\alpha)nt})^{\frac{1}{1-\alpha}},$$

(12) $$Y(t) = Y_0^* e^{nt}(1 + \mathrm{A}e^{-(1-\alpha)nt})^{\frac{\alpha}{1-\alpha}}.$$

1 We have $K_0^* = (s/n)^{\frac{1}{1-\alpha}} L_0$, $Y_0^* = (s/n)^{\frac{\alpha}{1-\alpha}} L_0$, and $\alpha^* = \alpha$ in this case.

Since $\lim_{t\to\infty}(1+\mathrm{A}e^{-(1-\alpha)nt})=1$, the *percentage* deviations of $K(t)$ and $Y(t)$ from the balanced growth paths $K_0^* e^{nt}$ and $Y_0^* e^{nt}$ tend to zero. The growth rate of the *absolute* difference is

(13) $$\frac{\dot{K}^*(t)-\dot{K}(t)}{K^*(t)-K(t)} = n + \mathrm{An}\frac{\left(1+\mathrm{A}e^{-(1-\alpha)nt}\right)^{\frac{\alpha}{1-\alpha}}}{e^{(1-\alpha)nt}\left[1-\left(1+\mathrm{A}e^{-(1-\alpha)nt}\right)^{\frac{1}{1-\alpha}}\right]}\,.$$

According to l'Hospital's rule, for all real numbers a and b $\lim_{t\to\infty} e^{at}\left[1-\left(1+\mathrm{A}e^{-at}\right)^{b}\right] = -\mathrm{Ab}$, therefore

(14) $$\lim_{t\to\infty} w_{K^*(t)-K(t)} = n-(1-\alpha)n = \alpha n\,.$$

The same applies for $Y^*(t)-Y(t)$.

The time path of L and K can be represented in a (L,K)-diagram. The balanced growth path runs on the straight line through the origin with slope k^*. If $K_0 \neq k^* L_0$, the deviation of K from the equilibrium path becomes larger with increasing L and diverges to infinity, as shown. Hence the time path of L and K does not converge to the balanced growth path, as shown in Solow's figure VI (here figure 2), but rather deviates even more from it as shown in figure 5. This does not contradict the convergence of capital intensity to its equilibrium value k^*. In the (L,K)-diagram this convergence means the following: if $K_0 < k^* L_0 (K_0 > k^* L_0)$, then for any $k < k^*$ $(k > k^*)$ there exists a time $t(k)$, so that after $t(k)$ the time path of L and K runs above (below) the straight line through the origin with slope k.

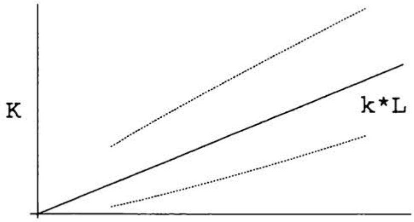

Figure 5

3. Endogenous Saving Decision

We have seen that the assumption of a constant saving rate in the Solow model ensures the divergence of $Y^*(t) - Y(t)$ (and $K^*(t) - K(t)$). This assumption is only a plausible first simplification. Assume now that at every time t the saving rate is chosen to

$$\text{maximize} \int_0^\infty e^{rt} u[c(t)]dt \,. \tag{15}$$

To see under which conditions $Y^*(t) - Y(t)$ then diverges, assume the Cobb-Douglas production function $Y = K^\alpha L^{1-\alpha}$ and the instantaneous utility function with constant risk aversion $u(c) = c^{1-\gamma}/(1-\gamma)$. It is well known that the per capita variables of the economy converge to a steady state described by $k^* = [\alpha/(n+r)]^{1/(1-\alpha)}$ and $c^* = k^{*\alpha} - nk^*$. For the absolute values $Y(t)$ and $Y^*(t)$ we have

Proposition 2:

The difference $Y^*(t) - Y(t)$ diverges if and only if

$$\gamma > (1-\alpha)\frac{r + n(1-\alpha)}{\alpha n} \,. \tag{16}$$

Proof:

Equation (7) shows that the long run growth rate of $Y^*(t) - Y(t)$ and the convergence rate λ of $y(t)$ add up to n, hence, $Y^*(t) - Y(t)$ diverges if and only if $\lambda < n$. In Blanchard / Fischer (1989, p. 47) λ is given as

$$\lambda = \frac{-r + \sqrt{r}^2 - 4y''(k^*)c^*/\gamma}{2} \,. \tag{17}$$

hence, $\lambda < n$ is equivalent to (16). q.e.d.

To understand the intuition behind the proposition, assume that $k(0) < k^*$. As long as $k(t) < k^*$, the higher marginal productivity of capital (compared to the steady state) gives an incentive to save more than in the steady state. This tends to reduce $Y^*(t) - Y(t)$. If the risk aversion γ is equal to zero, it is optimal to save all production until a finite time T in which the steady state is reached. In this case, $Y^*(t) - Y(t)$ is zero after T. If $\gamma > 0$, there is an additional contrary effect. There is an incentive to save less than in the steady state if $k(t) < k^*$ because there is a preference for a flat consumption path. This effect tends to enlarge $Y^*(t) - Y(t)$. This second effects dominates if γ is large enough to satisfy condition (16). Note that (16) contains the case $\gamma \geq (n+r)/(\alpha n)$. In that case the saving rate $s(t)$ is a constant or increasing function of $k(t)$, hence it is clear already from section 2 that $Y^*(t) - Y(t)$ diverges. Of course the conditions for divergent $Y^*(t) - Y(t)$ or increasing $s(t)$ are slightly different if one uses a different welfare criterion, e.g. max $\int_0^\infty e^{-rt} L(t) C(t)^{1-\gamma}/(1-\gamma) dt$.

4. Growth and Pollution

As long as one is mainly interested in per capita values, the divergence of $Y^* - Y$ is less important. However, when considering pollution not only the per capita value is relevant but also the absolute value. Net emissions growing without bound are certainly intolerable, even if they grow at a smaller rate than labor and production, i.e. if net emissions per head tend to zero. However, the instability of the balanced growth path in absolute terms as described above may cause exactly this outcome in an extension of the Solow model with pollution. We model the influence of the economy on the environment as the model introduced by Strøm (1973) in the simplified version of Bender (1976). Total emissions $bY(t)$ are proportional to production Y, where Y is a function of labor L and capital devoted to production, K_p. There is an abatement technology available, and total abatement $hK_u(t)$ is proportional to capital devoted to abatement, K_u. (Denote the per capita values of K_p, K_u and W by k_p, k_u and w respectively.) Hence, net emissions are given by

$$W(t) = bY(t) - hK_u(t) \,. \tag{18}$$

As in the Solow model we assume that the saving rate for productive capital, s_p, is constant:

$$\dot{K}_p(t) = s_p Y(t) \,. \tag{19}$$

We assume $0 < s_p < 1 - nb/h$, in particular $nb/h < 1$. We will investigate the two following rules for the accumulation of abatement capital[2]:

$$\dot{K}_u(t) = nb/hY^*(t) \,, \tag{20a}$$

$$\dot{K}_u(t) = nb/hY(t) \,. \tag{20b}$$

(20a) implies constant per-capita investments in abatement capital, (20b) implies a constant saving rate s_u for abatement capital, making this as-

[2] Of course the accumulation of abatement capital may influence s_p.

sumption formally very close to the Solow model. A clean environment is a public good, hence, as Buchholz and Cansier (1980) have already discussed, there is no market mechanism to ensure the accumulation of abatement capital as is the case for productive capital. Therefore, environmental policy has to ensure the accumulation of abatement capital by interfering with the market. One possibility is raising taxes and spending the proceeds for the accumulation of abatement capital. (20a) would then describe a constant poll tax nb/hY_0^*, (20b) a proportional income tax with constant tax rate nb/h. These simple tax schemes are motivated by the following observation: suppose first that the economy is on the balanced growth path $Y(t) = Y^*(t) = Y_0^* e^{nt}$ associated with s_p. A balanced growth path with constant net emissions is called a steady state. According to equation (18), $Y^*(t)$ is a steady state if and only if $\dot{K}_u(t) = b/h\dot{Y}^*(t)$, i.e. if abatement capital is accumulated as described in (20a) as well as in (20b). Hence, (20a) and (20b) describe the only poll tax and the only proportional income tax respectively (or, more generally speaking, the only constant per-capita abatement investments and the only constant s_u) capable of maintaining a steady state. Note that in a steady state k_p is constant at k_p^*, determined by $k_p^* = s_p/ny(k_p^*)$, k_u converges to $k_u^* = b/hy(k_p^*)$, and w converges to zero. If the economy is not on the balanced growth path, the two policy rules differ, but under both rules k_p, k_u and w converge to k_p^*, k_u^* and 0 respectively. Hence, under both rules not only is a steady state maintained, but also all per capita values converge to their steady state values. Therefore, as far as per capita values are concerned, under both policy rules a steady state can be called "stable". We want to show that nevertheless under both rules depending on the initial conditions net emissions in absolute terms can diverge to infinity because of the divergence of $Y^* - Y$.

Abatement capital and net emissions in absolute terms under (20a) are given by[3]

(21a) $$K_u(t) = K_u(0) + b/hY^*(t) - b/hY_0^*$$

(22a) $$W(t) = b[Y(t) - Y^*(t)] - hK_u(0) + bY_0^*$$

respectively and under (20b) by

(21b) $$K_u(t) = K_u(0) - nb/h \int_0^t [Y^*(\tau) - Y(\tau)]d\tau + b/hY^*(t) - b/hY_0^*$$

(22b) $$W(t) = -b[Y^*(t) - Y(t)] + nb \int_0^t [Y^*(\tau) - Y(\tau)]d\tau - hK_u(0) + bY_0^* .$$

[3] Note that $\int_0^t Y^*(\tau)d\tau = \int_0^t Y_0^* e^{n\tau} d\tau = \frac{1}{n}[Y^*(t) - Y_0^*]$.

Hence, the behavior of $W(t)$ is dominated by the behavior of the difference $Y(t) - Y^*(t)$. Equation (22a) immediately shows that under policy rule (20a) net emissions tend to infinity if $k_p(0) > k_p^*$ because of the divergence of $Y(t) - Y^*(t)$ derived in section 2.

Under policy rule (20b) we have two opposite effects: if $k_p(0) < k_p^*$, for all $t > 0$ there is less production than in the steady state, hence less emissions are caused by production, but also less abatement capital is accumulated, which then causes higher net emissions than in the steady state. Because of the divergence of the difference $Y^*(t) - Y(t)$ derived in section 2, both effects tend to infinity. Therefore, to determine the long run behavior of $W(t)$ we have to know even more about $Y^*(t) - Y(t)$. For the case of a Cobb-Douglas production function, $W(t)$ can be explicitly calculated. Inserting (12) into (22b) yields[4]

$$\text{(23b)} \quad W(t) = bY_o^*\left\{e^{nt}\left[1 + \mathrm{A}e^{-(1-\alpha)nt}\right]^{\frac{\alpha}{1-\alpha}} - e^{nt}\left[1 + \mathrm{A}e^{-(1-\alpha)nt}\right]^{\frac{1}{1-\alpha}} + [1 + \mathrm{A}]^{\frac{1}{1-\alpha}}\right\} - hK_u(0)\,.$$

l'Hospital's rule shows that if $A < 0$, i.e. if $k_p(0) < k_p^*$, $W(t)$ tends to infinity and the growth rate of $W(t)$ tends to αn.

We see that because of the divergence of $Y^*(t) - Y(t)$ associated with the constant s_p, under both policy rules net emissions may tend to infinity even if all per capita variables tend to their steady state values. As shown in section 3 individual utility maximization which does not take into account the external effects of production may cause the same outcome. Of course one can find more complicated environmental policy rules – depending on absolute values of some variables – which ensure that the economy converges to a steady state. What makes the policy rules (20a) and (20b) particularly interesting in the context of this paper is the reason for the failure to reach constant net emissions: the results concerning the divergence of $Y^*(t) - Y(t)$ derived in section 2. Even though these rules do not take into account $W(t)$ or any other absolute value, they would reach bounded $W(t)$ if $Y^*(t) - Y(t)$ were bounded or converged to zero sufficiently fast, respectively. So without these results one could not exclude them as unsuitable policy rules.

The same applies if one considers a model of optimal growth with pollution, where equation (18) holds and instantaneous utility depends on W or the accumulated stock of pollution P. If a central planer chooses investment in production and abatement capital as to maximize $\int_0^\infty e^{rt}u(c, W, P)dt$, the resulting time path, under suitable conditions, converges to a steady state as described above. The above discussion shows that setting the saving rate in productive capital equal to its long run value and following either (20a)

[4] Note that $\int e^{nt}\left[1 - \left(1 + \mathrm{A}e^{-(1-\alpha)nt}\right)^{\frac{\alpha}{1-\alpha}}\right]dt = \frac{1}{n}e^{nt} - \frac{1}{n}\left[e^{(1-\alpha)nt} + \mathrm{A}\right]^{\frac{1}{1-\alpha}}$.

or (20b) is not a suitable approximation for the optimal path outside the steady state. The very reason for this is the instability of the Solow model in absolute terms derived in section 2.

More generally speaking, whenever absolute values like W matter to people, the divergence of $Y^* - Y$ associated with a constant saving rate makes it much more important to take into account absolute rather than per capita values in decisions than it would otherwise be.

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Zusammenfassung

Im Wachstumsmodell von Solow konvergieren Kapitalintensität und Pro-Kopf-Einkommen bekanntlich gegen ihren jeweiligen Gleichgewichtswert. Dieser Beitrag zeigt, daß dagegen die Differenz zwischen gleichgewichtigem und tatsächlichem Wachstumspfad der Absolutgrößen im allgemeinen gegen unendlich divergiert. Er zeigt, wie diese Divergenz mit der Konvergenzrate der Pro-Kopf-Größen zusammenhängt, und beschreibt die Bedingungen, unter denen sie auftritt, falls die Sparentscheidung endogen ist. Ferner wird eine Implikation des Resultats für eine Erweiterung des Modells um Umweltverschmutzung dargestellt.

Abstract

In the Solow growth model, capital intensity and per capita income converge to their equilibrium values. This note shows that nevertheless the difference between the balanced and the actual growth path of the absolute values generally diverges to infinity. The paper shows how this divergence is related to the convergence rate for the per capita values and describes the conditions under which the result occurs if the saving decision is endogenous. An implication of the result is demonstrated in an extension of the model containing environmental pollution.

JEL Klassifikation: O41