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Uncertain Incomes, Pay-as-you-go Systems, and Diversification*

By Nils Hauenschild**

1. Introduction

In the last decade, the efficiency of public pension systems has been one of the major fields of research with respect to social security. Within a simple model of a small open economy it has been analyzed which system is (Pareto-)superior to the other and under which conditions a Pareto-improving transition from pay-as-you-go (henceforth PAYG) to fully funded social security is possible (see among others Breyer (1989), Homburg (1990), Breyer, Straub (1993), Brunner (1994), Fenge (1995) and Casarico (1998)). Perhaps surprisingly, those analyses have been largely confined to deterministic models. There are only a few papers dealing with the effects of social security in the presence of uncertain incomes and interest rates, most of them being concerned with the problem of intergenerational risk sharing (see Enders, Lapan (1982, 1993), Merton (1983), Gordon, Varian (1988) and Thøgersen (1998)). In that case, a PAYG pension system can be Pareto-improving because it spreads labour income risk between different generations and thereby eliminates an inefficiency of the capital market where such intergenerational risk sharing cannot be provided. Formally, this result crucially depends on the assumption of *unconditional* expectations that do not distinguish individuals by the state of nature they are born into and thus imply a judgement of social security systems *before* the birth of the individuals. Other authors (Gale, 1991, Richter, 1993) have adopted *conditional* expectations that judge pension systems only *after* the realization of the first period's shocks and thereby take all burdens and benefits resulting from those shocks into account. In these models there is no intergenerational risk sharing and a PAYG system is generally *not* Pareto-improving. A common feature of those two approaches is, however, that they do not use the standard model of a small open economy which limits, to some extent, the comparability with the results obtained in deterministic models.

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It is the purpose of this paper to analyze the impact of social security systems on expected utility within the model of a small open economy with exogenous and stochastic wages and interest rates. In this framework, the internal rates of return both of a fully funded and of a PAYG system are stochastic. Applying the criterion of conditional Pareto-optimality, which is based on the adoption of conditional expectations, it will be shown that a PAYG system can be Pareto-improving even though there is no intergenerational risk sharing. This result even holds if the PAYG system provides the lower expected rate of return and is both valid for the introduction of a "new" as well as the extension of an existing PAYG system. The economic intuition behind this result is that the PAYG system acts as a means of diversification because it can be regarded as an additional asset which is not perfectly correlated with the interest rate. Thus, the individuals would be willing to "invest" in the PAYG system voluntarily so as to optimize their saving portfolios. This diversificational effect cannot be interpreted as intergenerational risk sharing since each individual wants to participate in the PAYG system *after* observing its own labour income. The study thereby provides an argument against the complete abolition of unfunded pension systems as discussed in the economic literature but indicates that a "pension mix" with funded and unfunded components might be (Pareto-)efficient.

The paper is organized as follows. In section 2 we present the basic model and derive the conditional Pareto criterion. The implications of introducing respectively extending a PAYG system for expected utility are discussed in section 3. Some concluding remarks are presented in section 4.

2. The model

2.1 A stochastic small open economy

Consider an overlapping generations model of a small open economy with exogenously given wages and interest rates. Over time, the real wages $\{\widetilde{w}_t\}$ and the real interest rates $\{\widetilde{r}_t\}$ both follow a stochastic process.[1] In each period t, the random variables $\widetilde{w}_t$ and $\widetilde{r}_t$ are assumed to have a continuous density function and a compact and positive support. Note that they are *not* assumed to be independent over time or from each other. The economy consists of many identical individuals that live for only two periods. In his first period of life, a representative individual works and receives the real wage $\widetilde{w}_t$. In addition, he has to pay a contribution $\tau\widetilde{w}_t$ to a public pension system,

[1] Throughout the paper, we denote all exogenous random variables with a $\sim$ and their respective realizations with the same letter without a $\sim$.

where $0 \leq \tau < 1$ denotes the exogenous contribution rate. For $\tau = 0$ the model without social security is included as a special case. Because of the well-known neutrality property of fully funded pension systems, this is equivalent to the case that a fully funded system is established in the economy. In the second period, the individual is retired and receives no labour income. Consumption in that period consists of the previous period's savings s_t (including interest payments $\widetilde{r}_{t+1} \cdot s_t$) and of a pension whose amount depends on the established pension system. In a PAYG system the contributions are used to finance the pension payments for the old generation in the same period. Assuming a constant population growth rate $n(> -1)$ and imposing a balanced budget condition for the pension system in every period, the payments are thus given by $(1+n)\tau\widetilde{w}_{t+1}$. Again, the fully funded system, in which the contributions are invested and paid back during retirement, is included for $\tau = 0$ because of its neutrality property. Thus, we will not consider the fully funded system explicitly in the subsequent analysis but simply compare the PAYG system with the model without social security.

The individuals have a time-additive utility function with an instantaneous utility function u that is of the exponential form $u(c) = -(1/\beta) \cdot \exp(-\beta c), \beta > 0$, for all $c \in \mathbb{R}_+$. This utility function exhibits the property of constant absolute risk aversion (henceforth CARA) and is frequently used in models with uncertain incomes (see e.g. Caballero (1990, 1991)). Here, it is assumed to derive some analytical results in the subsequent sections (cf. equation (10)).[2] The individuals maximize expected utility conditional on the information available at time t, i.e. after observing their labour income w_t, subject to the intertemporal budget constraints stated above. Moreover, they are assumed to know the stochastic distributions of $\widetilde{w}_{t+1}$ and $\widetilde{r}_{t+1}$ exactly. Hence, they have to solve the optimization problem

$$(1) \qquad U(c_{1,t}, c_{2,t+1}) := u(c_{1,t}) + \frac{1}{1+\theta} E_t\left[u(c_{2,t+1})\right] \rightarrow \max_{c_{1,t} c_{2,t+1}} !$$

subject to

$$(2) \qquad c_{1,t} = w_t - s_t - \tau w_t \ ,$$

$$(3) \qquad c_{2,t+1} = (1 + \widetilde{r}_{t+1}) s_t + (1+n)\tau \widetilde{w}_{t+1} \ ,$$

where

$$(4) \qquad u(c) = -\frac{1}{\beta} e^{-\beta c}$$

[2] The qualitative results obtained below will, however, most presumably hold for more general utility functions as well, see the corresponding discussion at the end of section 3.1.

for all $c \in \mathbb{R}_+$. In (1) – (4), $\theta \geq 0$ denotes the rate of time preference, $\beta > 0$ the coefficient of absolute risk aversion, E_t the conditional expectation operator, and $c_{1,t}$ and $c_{2,t+1}$ are consumption of a young individual in period t and an old individual in period $t+1$, respectively. The necessary (and sufficient) optimality condition for (1) – (3) is given by

$$u'(c_{1,t}) = \frac{1}{1+\theta} E_t\left[(1+\widetilde{r}_{t+1})u'(c_{2,t+1})\right] . \tag{5}$$

Since the CARA utility function (4) does not fulfill the Inada conditions, we have to assume the existence of positive solutions $c^*_{1,t}$ and $c^*_{2,t+1}$ of (5). In view of the period length this seems to be an innocent assumption which does, however, *not* exclude optimal negative savings in the presence of a PAYG system.

In order to analyze the implications of social security systems for expected utility we have to derive the indirect utility function that gives maximum expected utility as a function of the contribution rate τ. For this purpose let $c^*_{1,t} > 0$, $c^*_{2,t+1} > 0$ and $s^*_t = w_t - \tau w_t - c^*_{1,t}$ denote the optimal values of consumption and savings resulting from (5), and denote by $V_t(\tau)$ indirect expected utility in dependence of τ. The function $V_t(\tau)$ is obtained by inserting the optimal consumption levels into (1), i.e.

$$V_t(\tau) = U(c^*_{1,t}, c^*_{2,t+1}) = u(c^*_{1,t}) + \frac{1}{1+\theta} E_t\left[u(c^*_{2,t+1})\right] . \tag{6}$$

2.2 The conditional Pareto criterion

In section 3 we are going to analyze whether it is (Pareto-)efficient to introduce respectively extend a PAYG system, i.e., we want to know if raising the contribution rate τ also raises expected utility of all generations. As already mentioned in the introduction, we will apply the concept of conditional Pareto-optimality to answer this question. This criterion is a natural extension of the standard Pareto-optimality to stochastic environments which simply substitutes the indirect (deterministic) utility function by expected utility (see Peled, 1982, p. 260). It is called *conditional* Pareto-optimality because it adopts conditional expectations that distinguish individuals by the state of nature they are born into and thus values utility conditional on the realization of the first period's random variables. Alternatively, one could use *unconditional* expectations that consider all individuals as equal and value utility before their births, thus not taking into account burdens and benefits of any policy measure that are due to the realization of the first period's random variables. With respect to efficiency, however, conditional expectations are more appropriate for three reasons. First, it can be argued that individual differences resulting from the state of nature should be treated just as differences in endowments and preferences, which is ensured by conditional expectations only (Peled, 1982, p. 270). Second, the fact that individuals who are not alive at the same time cannot draw risk sharing contracts with each other must not be interpreted as a capital market imperfection (Richter, 1993, p. 93). Consequently, there is no need to remove any imperfection by intergenerational risk sharing as induced by unconditional expectations (see e.g. Gordon, Varian, 1988). In view of those arguments, unconditional expectations seem to be more appropriate with respect to issues of intergenerational fairness and redistribution instead. Third, we have already used conditional expectations in the individuals' optimization problem which implies that all allocations are judged exactly the same as they would be judged by the individuals. Hence, the efficiency criterion is consistent with individual preferences.

Definition. Let $(c^A_{1,t}, c^A_{2,t+1})$, $t = 0, 1, 2, \ldots$, *denote the stochastic process of optimal consumption of all generations in the presence of a public pension system* A, *and let* $(c^B_{1,t}, c^B_{2,t+1})$, $t = 0, 1, 2, \ldots$, *denote the corresponding process for another system B. The public pension system A is called* conditional Pareto-superior *over system B, if for all* $t = 0, 1, 2, \ldots$

$$E_t\left[u(c^A_{1,t}) + \frac{1}{1+\theta}u(c^A_{2,t+1})\right] \geq E_t\left[u(c^B_{1,t}) + \frac{1}{1+\theta}u(c^B_{2,t+1})\right] \tag{7}$$

almost surely (a.s.), and if (7) is valid with strict inequality on some set with positive measure for at least one t^*.

The pension system A is called Pareto-optimal *if there is no other conditional Pareto-superior pension system.*

Let us now consider the indirect utility function according to (6) that gives maximum expected utility as a function of τ, *conditional* on the information in period t, i.e. conditional on the wage w_t. If $dV_t(\tau)/d\tau > 0$ for given w_t, that generation will be better off if the PAYG system is extended.[3] In view of the above definition we thus see that an extension of the PAYG system by raising τ is a (conditional) Pareto-improvement if $dV_t(\tau)/d\tau > 0$ for almost all realizations of $\widetilde{w}_t$ and for all t.[4] If, on the other hand, $dV_t(\tau)/d\tau < 0$ with positive probability for at least one t, then a (marginal)

[3] This approach is basically the same as in Richter (1993), though he only considers $dV_t(\tau)/d\tau$ at $\tau = 0$, i.e. the introduction of a PAYG system.

[4] In this context, the role of the pension system B in the definition is adopted by the fully funded system respectively the model without social security.

extension of the PAYG system is no Pareto-improvement. In this context, note that $dV_t(\tau)/d\tau < 0$ a.s. for all t does *not* imply that a reduction of τ is Pareto-improving because the old generation of the period in which τ is reduced receives a lower pension and is thus worse off. Finally, we are able to judge whether the *introduction* of a PAYG system into an economy without social security is Pareto-improving if we consider $dV_t(\tau)/d\tau$ at $\tau = 0$.

In order to determine the sign of $dV_t(\tau)/d\tau$, we have to start with $V_t(\tau)$ as been given by (6). Applying the envelope theorem to this equation yields

$$\text{(8)} \qquad \begin{aligned} \frac{dV_t(\tau)}{d\tau} &= \frac{\partial U(c_{1,t}, c_{2,t+1})}{\partial \tau} \bigg|_{(c_{1,t}, c_{2,t+1}) = (c^*_{1,t}, c^*_{2,t+1})} \\ &= u'(c^*_{1,t}) \cdot (-w_t) + \frac{1}{1+\theta} E_t\big[u'(c^*_{2,t+1}) \cdot (1+n)\widetilde{w}_{t+1}\big] \,. \end{aligned}$$

Combining (8) with the optimality condition (5) then gives

$$\text{(9)} \qquad \begin{aligned} \frac{dV_t(\tau)}{d\tau} &= \frac{1}{1+\theta}\Big(-w_t E_t\Big[(1+\widetilde{r}_{t+1})u'\big(c^*_{2,t+1}\big)\Big] + E_t\Big[(1+n)\widetilde{w}_{t+1}u'\big(c^*_{2,t+1}\big)\Big]\Big) \\ &= \frac{1}{1+\theta} E_t\Big[\big((1+n)\widetilde{w}_{t+1} - (1+\widetilde{r}_{t+1})w_t\big) \cdot u'(c^*_{2,t+1})\Big] \,. \end{aligned}$$

Unfortunately, the sign of the conditional expectation in (9) is not unique and cannot be determined for arbitrary utility functions and distributions of $\widetilde{w}_{t+1}$ and $\widetilde{r}_{t+1}$. At this point we can, however, make use of the specific CARA utility function assumed above, i.e. (9) takes on the form[5]

$$\text{(10)} \qquad \frac{dV_t(\tau)}{d\tau} = \frac{1}{1+\theta} E_t\Big[\big((1+n)\widetilde{w}_{t+1} - (1+\widetilde{r}_{t+1})w_t\big) \cdot e^{-\beta((1+\widetilde{r}_{t+1})s^*_t + (1+n)\tau\widetilde{w}_{t+1})}\Big] \,.$$

Let us denote by $\overline{w}_{t+1}$ and $\sigma^2_{w_{t+1}}$ respectively $\overline{r}_{t+1}$ and $\sigma^2_{r_{t+1}}$ the (conditional) expectations and variances of $\widetilde{w}_{t+1}$ and $\widetilde{r}_{t+1}$, and their covariance by $\sigma_{w_{t+1}r_{t+1}}$. Using a second order Taylor approximation of (10) around the expectations $\overline{w}_{t+1}$ and $\overline{r}_{t+1}$ yields an expression of $dV_t(\tau)/d\tau$ that only depends on those expectations and the variances. It allows us to show in detail how the uncertainty in wages and interest rates affects expected utility. The derivation of the Taylor approximation is rather tedious and only presented in the appendix. Ignoring terms of third and higher order we obtain

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[5] As already indicated in section 2.1, some potential consequences of the choice of that utility function will be discussed below.

$$
\begin{aligned}
(11)\quad \frac{dV_t(\tau)}{d\tau} = & \frac{1}{1+\theta} w_t \exp\Big(-\beta\big((1+\overline{r}_{t+1})s_t^* + (1+n)\tau\overline{w}_{t+1}\big)\Big) \\
& \times\Bigg\{\triangle_t + \beta s_t^*\left(\sigma^2_{r_{t+1}} - \frac{1+n}{w_t}\sigma_{w_{t+1}r_{t+1}}\right) + \beta(1+n)\tau\left(\sigma_{w_{t+1}r_{t+1}} - \frac{1+n}{w_t}\sigma^2_{w_{t+1}}\right) \\
& + \frac{1}{2}\triangle_t\,\beta^2\Big((s_t^*)^2\sigma^2_{r_{t+1}} + 2s_t^*(1+n)\tau\sigma_{w_{t+1}r_{t+1}} + (1+n)^2\tau^2\sigma^2_{w_{t+1}}\Big)\Bigg\},
\end{aligned}
$$

where $\triangle_t := (1+n)\frac{\overline{w}_{t+1}}{w_t} - (1+\overline{r}_{t+1})$. In the appendix it is shown that the sum in brackets of the last term in (11) simplifies to

$$
(12)\qquad \sigma^2_{c_{2,t+1}} := Var_t(c^*_{2,t+1}) = Var_t\big((1+\widetilde{r}_{t+1})s_t^* + (1+n)\tau\widetilde{w}_{t+1}\big) \geq 0 .
$$

Furthermore, note that the internal rate of return $\widetilde{z}_{t+1}$ of the PAYG system is defined by

$$
(13)\qquad 1+\widetilde{z}_{t+1} = (1+n)\frac{\widetilde{w}_{t+1}}{w_t} .
$$

Hence, $\triangle_t$ gives the difference of the expected rates of return of a PAYG and a fully funded system, the latter being equal to the expected interest rate and thus the rate of return on private savings. Of course, $\triangle_t$ depends on the real wage w_t that determines the contributions payable to the pension system. The variance $\sigma^2_{z_{t+1}}$ of $\widetilde{z}_{t+1}$ and its covariance $\sigma_{z_{t+1}r_{t+1}}$ with the interest rate have the properties $\frac{1+n}{w_t}\sigma_{w_{t+1}r_{t+1}} = \sigma_{z_{t+1}r_{t+1}}$ as well as $\left(\frac{1+n}{w_t}\right)^2\sigma^2_{w_{t+1}} = \sigma^2_{z_{t+1}}$. With these notations, (11) can equivalently be written as

$$
\begin{aligned}
(14)\quad \frac{dV_t(\tau)}{d\tau} = & \frac{1}{1+\theta} w_t \exp\Big(-\beta\big((1+\overline{r}_{t+1})s_t^* + (1+n)\tau\overline{w}_{t+1}\big)\Big) \\
& \times\left\{\triangle_t\left(1+\frac{1}{2}\beta^2\sigma^2_{c_{2,t+1}}\right) + \beta s_t^*\left(\sigma^2_{r_{t+1}} - \sigma_{z_{t+1}r_{t+1}}\right) + \beta\tau w_t\cdot\left(\sigma_{z_{t+1}r_{t+1}} - \sigma^2_{z_{t+1}}\right)\right\}.
\end{aligned}
$$

We are now in position to show when the introduction or extension of a PAYG system is Pareto-improving.

3. Diversification and Pareto-improvements

3.1 Introducing a pay-as-you-go system

Let us first consider the case that a PAYG system may be introduced into an economy without any social security system in an arbitrary period t_0. The (marginal) introduction of a PAYG system is a Pareto-improvement if $dV_t(\tau)/d\tau\mid_{\tau=0} > 0$ a.s. for all $t \geq t_0$, i.e., with respect to (14), if

$$\triangle_t \cdot \left(1 + \frac{1}{2}\beta^2 \sigma^2_{c_{2,t+1}}\right) + \beta s^*_t \cdot \left(\sigma^2_{r_{t+1}} - \sigma_{z_{t+1} r_{t+1}}\right) > 0 \tag{15}$$

almost surely for all $t \geq t_0$. The introduction is no Pareto-improvement if (15) holds with the opposite inequality with positive probability in one period $t \geq t_0$.

A few remarks are necessary with respect to equation (15). First, it is noteworthy that the variance of the internal rate of return $\widetilde{z}_{t+1}$ of the PAYG system does not enter the criterion. Though this seems to be surprising in the first place, it becomes plausible at remembering that we only consider a *marginal* introduction of a PAYG system. Hence, there is yet no "real" dependence on the stochastic wages. The marginally changing risk structure is represented by the covariance with $\widetilde{r}_{t+1}$ instead. Second, the first term in (15) only favours the PAYG system if it provides the higher expected rate of return. In case of no uncertainty, i.e. $\sigma^2_{r_{t+1}} = \sigma_{z_{t+1} r_{t+1}} = \sigma^2_{c_{2,t+1}} = 0$, this fact thus reestablishes the corresponding result of the deterministic models cited in the introduction. For the subsequent analysis, finally note that $\triangle_t \neq 0$ with probability one since the distribution of $\widetilde{w}_t$ was assumed to have a continuous density function, and that the assumption of positive consumption in both periods implies $s^*_t > 0$ in case of $\tau = 0$.

From equation (15) only two unambiguous conclusions can be drawn. On the one hand, the PAYG system is a Pareto-improvement if $\triangle_t > 0$ a.s. for all t and if $\sigma_{z_{t+1} r_{t+1}} < \sigma^2_{r_{t+1}}$ for all t.[6] In this case, the PAYG system provides an expected internal rate of return that exceeds the expected interest rate and reduces the overall risk about second period's consumption of all generations. Hence, it is obvious that all generations are better off with that pension system. Since the old generation in period t_0 benefits from the usual initial gains, the PAYG system is clearly Pareto-improving. On the other hand, a PAYG system is *not* Pareto-improving if $\triangle_t < 0$ and $\sigma_{z_{t+1} r_{t+1}} > \sigma^2_{r_{t+1}}$ in at least on period $t \geq t_0$ with positive probability. Here, the PAYG system provides the lower expected rate of return and increases the risk this generation has to bear.

The situation is less clear-cut when the terms in equation (15) have opposite signs. If $\triangle_t > 0$, there is a strong tendency for the PAYG system to be Pareto-improving. It is only if the variance of the interest rate is "small" and the covariance between wages and interest rates is "large" that a generation will be worse off in the PAYG system. The reason is that this constellation also implies a large variance of the real wage (since the covariance cannot exceed both variances) such that a PAYG system burdens that gen-

[6] Note that $\triangle_t > 0$ a.s. is only possible because we have assumed compact and positive supports for the random variables.

eration with a large amount of additional risk that outweighs the advantage of a higher expected return. If, on the other hand, $\triangle_t < 0$ there is a tendency towards the PAYG system being *no* Pareto-improvement. Nevertheless, if the variance of the interest rate is "large" and the covariance between interest rates and wages is "small" or even negative, it is possible that every generation is better off in a PAYG system. This is because a PAYG system serves as a means of diversification. When there is no social security system, second period consumption fully depends on the risk associated with the uncertain interest rate. The introduction of a PAYG system makes second period consumption depend on the uncertain wage as well. In case of a small or negative covariance this reduces the risk associated with second period consumption and, from the individuals' point of view, offers the possibility to hold a diversified portfolio. If this diversificational effect is strong enough, all individuals are prepared to accept a lower expected rate of return, and the PAYG system can be Pareto-improving.

This result should be contrasted with the effect of intergenerational risk sharing. There, the income risk of yet *unborn* individuals is spread over different generations such that "unlucky" generations with low incomes benefit from the higher incomes of "lucky" generations. This potential benefit for all individuals counts as a Pareto-improvement in case of unconditional expectations even though the "lucky" generations could be worse off than in the absence of a PAYG system after observing their labour income. The diversificational effect described above is different since each individual can be made better off knowing about his labour income, i.e. *after* observing whether he is a "winner" or a "loser". Even lucky generations are willing to participate in the PAYG system voluntarily because they as well benefit from the diversified portfolio. The possibility of a conditional Pareto-improvement due to diversification thus establishes a result in favour of PAYG systems that is not based on any intergenerational risk sharing but only on efficiency grounds.

Before proceeding further, it seems appropriate to discuss how these results depend on our choice of the specific CARA utility function. As already indicated above, it was primarily chosen for analytical reasons since the analysis following equation (9) (i.e. the Taylor approximation) is rather intractable for other explicit or even general concave utility functions. It cannot be supposed, however, that the qualitative results obtained would change significantly in those cases. In order to illustrate this presumption, observe that the most relevant property with respect to the individuals' behaviour towards risk displayed by the CARA utility function is, of course, constant absolute risk aversion. It is frequently considered more plausible to assume *decreasing* absolute risk aversion (e.g. constant relative risk aversion), implying that the individuals are prepared to accept higher risks as they become wealthier. In that case, the first period income w_t will become a relevant determinant of the individuals' diversificational considerations as well. As an example, consider the case that $\triangle_t < 0$ but $\sigma^2_{r_{t+1}} > \sigma_{z_{t+1}r_{t+1}}$, i.e., a PAYG system provides the lower expected rate of return but reduces the risk of individual saving portfolios as their dependence on the risky interest rate is reduced. If w_t is high, an individual displaying decreasing absolute risk aversion will thus consider the PAYG system less attractive than an individual with constant absolute risk aversion, because the former is not concerned with the risky interest rates that much and hence does not benefit from the diversified portfolio as much as the latter one does. Analogously, if w_t is low, an individual with decreasing absolute risk aversion will consider a PAYG system *more* attractive than an individual with constant absolute risk aversion.[7] The main point, however, is that the possibility of opposite and compensating effects is still present (though their sizes will be different), such that the above conclusions remain qualitatively valid. In view of these considerations, the assumption of a CARA utility function does not appear too restrictive with respect to the relevant results.

3.2 Extending an established pay-as-you-go system

Let us now turn to the case that a PAYG system is already established in the economy and ask whether it is Pareto-improving to *extend* that system even further by raising τ. As already mentioned in section 2, a *reduction* of the contribution rate τ will be no Pareto-improvement even if all subsequent generations would be better off. It must be left for future research to analyze if the losses of the old generation resulting from a reduction of τ can be compensated for by subsequent generations so as to achieve a Pareto-improvement. Under certain conditions such a compensation is possible in deterministic models (see the references given in the introduction) as well as in a model with stochastic wages and *deterministic* interest rates (see Hauenschild, 1999a, b). Since this problem is beyond the scope of this paper we will concentrate on Pareto-improvements through diversification by raising τ, i.e., we consider equation (14) for $\tau > 0$.

The situation is a little more complicated than in the previous subsection because of the additional term and the possibility of negative savings. If the contribution rate τ is excessively high, individuals will lend upon their retirement income abroad at the interest rate $\widetilde{r}_{t+1}$ and thus have negative sav-

[7] Similar conclusions can be drawn for the other two cases discussed above. If both terms in (15) have the same sign, the results remain entirely unchanged. If $\triangle_t > 0$ and $\sigma^2_{r_{t+1}} < \sigma_{z_{t+1}r_{t+1}}$, high income generations will consider a PAYG system more attractive and low income generations will consider a PAYG system less attractive than in case of constant absolute risk aversion.

ings. We will, however, concentrate on the case $s_t^* > 0$ first. An extension of the PAYG system in a period t_0 is Pareto-improving if $\triangle_t > 0$ a.s. for all $t \geq t_0$, $\sigma^2_{r_{t+1}} > \sigma_{z_{t+1}r_{t+1}}$ for all $t \geq t_0$ and $\sigma_{z_{t+1}r_{t+1}} > \sigma^2_{z_{t+1}}$ for all $t \geq t_0$. Here, the PAYG system both offers the higher expected rate of return and reduces the overall risk by shifting more weight to the less risky wages. On the other hand, the extension of a PAYG system is no Pareto-improvement if $\triangle_t < 0$, $\sigma^2_{r_{t+1}} < \sigma_{z_{t+1}r_{t+1}}$ and $\sigma_{z_{t+1}r_{t+1}} < \sigma^2_{z_{t+1}}$ in at least one period. Here, the PAYG system provides the lower expected rate of return and *increases* the risk for that specific generation.

If the terms in (14) have opposite signs (especially when $s_t^* < 0$), it is no longer possible to draw unambiguous conclusions. As in the previous subsection we can, however, point out several tendencies and especially show how the different terms affect the Pareto criterion. For this purpose, it is convenient to rearrange equation (14) and thereby obtain the equivalent criterion that an extension of the PAYG system is Pareto-improving if

$$(16) \qquad \triangle_t \cdot \left(1 + \frac{1}{2}\beta^2 \sigma^2_{c_{2,t+1}}\right) + \beta \cdot s_t^* \cdot \sigma^2_{r_{t+1}} - \beta \cdot \tau \cdot w_t \cdot \sigma^2_{z_{t+1}} + \beta(\tau w_t - s_t^*) \cdot \sigma_{z_{t+1}r_{t+1}} > 0$$

a.s. for all $t \geq t_0$. As usual, the first term in (16) only favours the PAYG system if it provides the higher expected rate of return. As long as $s_t^* > 0$, the term $\beta \cdot s_t^* \cdot \sigma^2_{r_{t+1}}$ is positive and favours the PAYG system as well. This becomes plausible if the "crowding out" effect of social security systems, i.e., the fact that raising the contribution rate τ induces a reduction in private savings, is taken into account. An extended PAYG system thus reduces the importance of private savings for second period consumption and hence the relevance of the risk associated with the interest rate. If the current contribution rate τ is so high that $s_t^* < 0$, the term $\beta \cdot s_t^* \cdot \sigma^2_{r_{t+1}}$ reduces expected utility when the PAYG system is further extended, because the individuals increase their negative savings (i.e. they take up higher debts), thereby *increasing* their dependence on the risky interest rates.

The uncertainty associated with the PAYG system itself enters the criterion (16) through the variance of the wage, i.e. via the term $\beta \cdot \tau \cdot w_t \cdot \sigma^2_{z_{t+1}}$. Of course it is not at all surprising that this term implies a reduction in expected utility if the PAYG system is further extended. Note that this effect is the stronger, the larger the current value of τ. The sign of the last term in (16) is ambiguous since it depends both on the covariance of wages and interest rates and on whether a larger part of savings for retirement is currently carried out over private savings (s_t^*) or over the established PAYG system (τw_t). A negative covariance thus affects the criterion (16) in favour of the PAYG system only if it currently has the lesser importance for retirement savings ($\tau w_t < s_t^*$). If the current value of τ is high, the factor $(\tau w_t - s_t^*)$

is positive and a further extension of the PAYG system reduces expected utility if $\sigma_{z_{t+1}r_{t+1}} < 0$. These results are perfectly consistent with the intuitive idea of a diversificational effect in case of negatively correlated assets. The reverse conclusions hold for $\sigma_{z_{t+1}r_{t+1}} > 0$.

Summing up the above considerations, the possibility of a Pareto-improvement by further extending an established PAYG system crucially depends on the current value of the contribution rate τ. The higher τ, i.e. the more extensive the present PAYG system, the less likely it is that extending the PAYG system will be beneficial to all subsequent generations. However, if the established PAYG system is not too excessive (i.e. τ is small), extending the PAYG system can be Pareto-improving even though it provides the lower expected rate of return. Just as in the previous subsection, this is due to diversificational considerations only and does not rely on any kind of intergenerational risk sharing.[8]

4. Concluding remarks

In this paper we have analyzed the efficiency of PAYG financed public pension systems within the model of a small open economy with stochastic wages and interest rates. Applying the criterion of conditional Pareto-optimality which is based on the adoption of conditional expectations, we have shown that a PAYG system can be Pareto-improving even though its expected rate of return is below the expected interest rate. This result holds in case of a favourable variance-covariance structure that implies a diversificational effect of a PAYG system. In the presence of such a system, retirement income depends both on the interest rate and on the wage of the next generation, and may thus be associated with less risk than without social security, where only the stochastic interest rate is relevant. Moreover, the result is not only true for the introduction of a PAYG system into an economy without social security but also holds with respect to the extension of an existing PAYG system. Here, the current value of the contribution rate τ is of crucial importance. The higher the initial value of τ, the more likely that a further extension of the PAYG system is no Pareto-improvement.

The findings of this study ought to be contrasted with several results recently derived in the literature. First, they differ from the one obtained by Richter (1993) who shows that the introduction of a PAYG system is *never* Pareto-improving if there is a long-lived asset, land. Since the author

[8] One can also replicate the considerations with respect to the CARA utility function made at the end of section 3.1. If there is decreasing absolute risk aversion, it is reasonable to expect that only the size but not the sign of the respective terms in (16) will change, thereby leaving the qualitative results unaltered.

adopts conditional expectations as well, it has to be concluded that this specific asset excludes diversificational effects like the ones described in this paper. Second, our findings are related to the ones presented by Gordon, Varian (1988) and Enders, Lapan (1982, 1993). They use unconditional expectations and show that PAYG systems are Pareto-improving because they provide intergenerational risk sharing. The above diversification effect is similar with respect to the conclusions, but the effects must not be identified with each other. Formally, this can be justified with the different form of expectations used in the optimization problem and in the efficiency criterion. The economic rationale is that intergenerational risk sharing means dividing future risks between future generations whereas diversification amounts to investing into the PAYG system voluntarily so as to optimize savings portfolios. Hence, it is not surprising that a PAYG system is not always a Pareto-improvement in the latter case, but only under certain conditions.

It can be concluded that this paper provides an argument in favour of PAYG systems which does not rely on any aspects of intergenerational fairness but is simply based on efficiency grounds and portfolio optimization. It is not claimed, however, that a PAYG system is always and everywhere beneficial, which is especially true for excessive pension systems with already high contribution rates. Moreover, the possibility of a Pareto-improvement might be restricted to rather special cases, since the inequalities (15) and (16) would have to hold *for all t*. Nevertheless, the above analysis shows that a PAYG system which provides an expected rate of return that is below the expected interest rate is not necessarily undesirable from the individuals' point of view. Instead, it has to be checked carefully whether there may be efficiency gains from PAYG systems through diversification and which is the "right" amount of social security, i.e. the contribution rate that "optimizes" individual saving portfolios.

Appendix

In this appendix we will at first present the derivation of the Taylor approximation used in (11). To simplify notations we omit all subscripts, i.e. $\widetilde{r} := \widetilde{r}_{t+1}$, $\widetilde{w} := \widetilde{w}_{t+1}$, $\overline{r} := \overline{r}_{t+1}$, $\overline{w} := \overline{w}_{t+1}$, $w := w_t$ and $s := s_t^*$.

Define

$$h(\widetilde{r}, \widetilde{w}) := \big((1+n)\widetilde{w} - (1+\widetilde{r})w\big) \cdot \exp\Big(-\beta\big((1+\widetilde{r})s + (1+n)\tau\widetilde{w}\big)\Big) ,$$

such that the r.h.s. of (10) equals $\frac{1}{1+\theta} E_t\big[h(\widetilde{r}, \widetilde{w})\big]$. For notational convenience let

$$g(\widetilde{r}, \widetilde{w}) := \exp\Big(-\beta\big((1+\widetilde{r})s + (1+n)\tau\widetilde{w}\big)\Big) .$$

A second order Taylor approximation of h around the expectations $\overline{r}$ and $\overline{w}$ is given by (terms of third and higher order are omitted)

$$h(\widetilde{r},\widetilde{w}) \approx h(\overline{r},\overline{w}) + (\operatorname{grad} h)(\overline{r},\overline{w}) \cdot \begin{pmatrix} \widetilde{r}-\overline{r} \\ \widetilde{w}-\overline{w} \end{pmatrix} + \frac{1}{2}(\widetilde{r}-\overline{r},\widetilde{w}-\overline{w}) \cdot (\operatorname{Hess} h)(\overline{r},\overline{w}) \cdot \begin{pmatrix} \widetilde{r}-\overline{r} \\ \widetilde{w}-\overline{w} \end{pmatrix}, \tag{17}$$

where

$$(\operatorname{grad} h)(\overline{r},\overline{w}) := \left(\frac{\partial h}{\partial \widetilde{r}}(\overline{r},\overline{w}), \frac{\partial h}{\partial \widetilde{w}}(\overline{r},\overline{w})\right)$$

and

$$(\operatorname{Hess} h)(\overline{r},\overline{w}) := \begin{pmatrix} \frac{\partial^2 h}{\partial \widetilde{r}^2}(\overline{r},\overline{w}) & \frac{\partial^2 h}{\partial \widetilde{r}\partial \widetilde{w}}(\overline{r},\overline{w}) \\ \frac{\partial^2 h}{\partial \widetilde{w}\partial \widetilde{r}}(\overline{r},\overline{w}) & \frac{\partial^2 h}{\partial \widetilde{w}^2}(\overline{r},\overline{w}) \end{pmatrix}$$

denote the gradient respectively the Hessian of h at $(\overline{r},\overline{w})$.

First, we compute the partial derivatives of h and obtain

$$\begin{aligned}
\frac{\partial h}{\partial \widetilde{r}} &= -w \cdot g(\widetilde{r},\widetilde{w}) + [(1+n)\widetilde{w} - (1+\widetilde{r})w](-\beta s) \cdot g(\widetilde{r},\widetilde{w}) , \\
\frac{\partial^2 h}{\partial \widetilde{r}^2} &= -w(-\beta s) \cdot g(\widetilde{r},\widetilde{w}) - w(-\beta s) \cdot g(\widetilde{r},\widetilde{w}) \\
&\quad + [(1+n)\widetilde{w} - (1+\widetilde{r})w](-\beta s)(-\beta s) \cdot g(\widetilde{r},\widetilde{w}) , \\
\frac{\partial^2 h}{\partial \widetilde{r}\partial \widetilde{w}} &= -w[-\beta(1+n)\tau] \cdot g(\widetilde{r},\widetilde{w}) + (1+n)(-\beta s) \cdot g(\widetilde{r},\widetilde{w}) \\
&\quad + [(1+n)\widetilde{w} - (1+\widetilde{r})w](-\beta s)[-\beta(1+n)\tau] \cdot g(\widetilde{r},\widetilde{w})
\end{aligned}$$

as well as

$$\begin{aligned}
\frac{\partial h}{\partial \widetilde{w}} &= (1+n) \cdot g(\widetilde{r},\widetilde{w}) + [(1+n)\widetilde{w} - (1+\widetilde{r})w] \cdot [-\beta(1+n)\tau] \cdot g(\widetilde{r},\widetilde{w}) , \\
\frac{\partial^2 h}{\partial \widetilde{w}^2} &= (1+n)[-\beta(1+n)\tau] \cdot g(\widetilde{r},\widetilde{w}) + (1+n)[-\beta(1+n)\tau] \cdot g(\widetilde{r},\widetilde{w}) \\
&\quad + [(1+n)\widetilde{w} - (1+\widetilde{r})w] \cdot [-\beta(1+n)\tau] \cdot [-\beta(1+n)\tau] \cdot g(\widetilde{r},\widetilde{w}) , \\
\frac{\partial^2 h}{\partial \widetilde{w}\partial \widetilde{r}} &= (1+n)(-\beta s) \cdot g(\widetilde{r},\widetilde{w}) + w\beta(1+n)\tau \cdot g(\widetilde{r},\widetilde{w}) \\
&\quad + [(1+n)\widetilde{w} - (1+\widetilde{r})w] \cdot [-\beta(1+n)\tau] \cdot (-\beta s) \cdot g(\widetilde{r},\widetilde{w}) .
\end{aligned}$$

Inserting into (17) yields

$$
\begin{aligned}
h(\widetilde{r},\widetilde{w}) \approx & \left[(1+n)\overline{w}-(1+\overline{r})w\right]\cdot g(\overline{r},\overline{w}) \\
& +\Big\{-w+\left[(1+n)\overline{w}-(1+\overline{r})w\right](-\beta s)\Big\}g(\overline{r},\overline{w})\cdot(\widetilde{r}-\overline{r}) \\
& +\Big\{(1+n)+\left[(1+n)\overline{w}-(1+\overline{r})w\right]\cdot\left[-\beta(1+n)\tau\right]\Big\}g(\overline{r},\overline{w})\cdot(\widetilde{w}-\overline{w}) \\
& +\frac{1}{2}\Big\{2w\beta s+\left[(1+n)\overline{w}-(1+\overline{r})w\right]\beta^2 s^2\Big\}g(\overline{r},\overline{w})\cdot(\widetilde{r}-\overline{r})^2 \\
& +\Big\{w\beta(1+n)\tau-(1+n)\beta s+[(1+n)\overline{w}-(1+\overline{r})w]\beta^2 s(1+n)\tau\Big\} \\
& \quad \cdot g(\overline{r},\overline{w})\cdot(\widetilde{r}-\overline{r})(\widetilde{w}-\overline{w}) \\
& +\frac{1}{2}\Big\{-2\beta(1+n)^2\tau+[(1+n)\overline{w}-(1+\overline{r})w]\beta^2(1+n)^2\tau^2\Big\} \\
& \quad \cdot g(\overline{r},\overline{w})\cdot(\widetilde{w}-\overline{w})^2 .
\end{aligned}
$$

Inserting this expression for h into

$$\frac{dV_t(\tau)}{d\tau}=\frac{1}{1+\theta}E_t\left[h(\widetilde{r},\widetilde{w})\right]$$

and taking $E_t\left[(\widetilde{r}-\overline{r})\right]=E_t\left[(\widetilde{w}-\overline{w})\right]=0$ into account gives

$$
\begin{aligned}
\frac{dV_t(\tau)}{d\tau}= & \frac{1}{1+\theta}w_t\exp\Big(-\beta\big((1+\bar{r}_{t+1})s_t^*+(1+n)\tau\overline{w}_{t+1}\big)\Big) \\
& \times\Bigg(\triangle_t+\frac{1}{2}\left[2\beta s_t^*+\triangle_t\beta^2(s_t^*)^2\right]\sigma^2_{r_{t+1}} \\
& +\left[\beta(1+n)\tau-(1+n)\beta\frac{s_t^*}{w_t}+\triangle_t\beta^2 s_t^*(1+n)\tau\right]\sigma_{w_{t+1}r_{t+1}} \\
& +\frac{1}{2}\left[-2\beta(1+n)^2\tau\frac{1}{w_t}+\triangle_t\beta^2(1+n)^2\tau^2\right]\sigma^2_{w_{t+1}}\Bigg) .
\end{aligned}
$$

Rearranging this equation immediately yields (11) given in the text.

In section 2.2 we have claimed that the factor

$$\left((s_t^*)^2\sigma^2_{r_{t+1}}+2s_t^*(1+n)\tau\sigma_{r_{t+1}w_{t+1}}+(1+n)^2\tau^2\sigma^2_{w_{t+1}}\right) \tag{18}$$

in (11) equals the variance of second period consumption according to (12). Using the abbreviations given at the beginning of the appendix, (18) can be written and transformed as follows:

$$s^2 E_t\left[(\widetilde{r}-\overline{r})^2\right] + 2s(1+n)\tau E_t[(\widetilde{r}-\overline{r})(\widetilde{w}-\overline{w})] + (1+n)^2\tau^2 E_t\left[(\widetilde{w}-\overline{w})^2\right]$$

$$= E_t\left[\left(s(\widetilde{r}-\overline{r})\right)^2 + 2s(\widetilde{r}-\overline{r})\cdot(1+n)\tau(\widetilde{w}-\overline{w}) + \left((1+n)\tau(\widetilde{w}-\overline{w})\right)^2\right]$$

$$= E_t\left[\left(s(\widetilde{r}-\overline{r}) + (1+n)\tau(\widetilde{w}-\overline{w})\right)^2\right]$$

$$= E_t\left[\left((1+\widetilde{r})s + (1+n)\tau\widetilde{w} - (1+\overline{r})s - (1+n)\tau\overline{w}\right)^2\right]$$

$$= E_t\left[\Big((1+\widetilde{r})s + (1+n)\tau\widetilde{w} - E_t\big[(1+\widetilde{r})s + (1+n)\tau\widetilde{w}\big]\Big)^2\right]$$

$$= Var_t((1+\widetilde{r})s + (1+n)\tau\widetilde{w})\ .$$

This proves the assertion.

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Zusammenfassung

Diese Arbeit analysiert die Effizienz des Umlageverfahrens im Rahmen des Modells einer kleinen offenen Volkswirtschaft mit stochastischen Löhnen und Zinsen. Unter Verwendung des Kriteriums der bedingten Pareto-Optimalität wird gezeigt, daß sowohl die Einführung als auch die Ausdehnung eines bestehenden Umlageverfahrens selbst dann eine Pareto-Verbesserung darstellen kann, wenn dessen erwarteter interner Zinssatz unterhalb des erwarteten Marktzinses liegt. Dieses Ergebnis basiert ausschließlich auf Effizienzüberlegungen und ergibt sich nicht aus einer möglichen intergenerationalen Risikoteilung. Es folgt alleine daraus, daß das Umlageverfahren wie ein Diversifikationsinstrument wirkt, indem es das Gesamtrisiko der individuellen Ersparnisportfolios reduziert.

Abstract

This paper analyzes the efficiency of pay-as-you-go pension systems in a small open economy with stochastic wages and interest rates. Applying the criterion of conditional Pareto-optimality it is shown that the introduction as well as the extension of an existing pay-as-you-go system can be Pareto-improving even though its expected rate of return is below the expected interest rate. This result is only based on efficiency grounds and not due to any intergenerational risk sharing. It follows from the fact that a pay-as-you-go system acts as a means of diversification by reducing the overall risk of individual saving portfolios.

JEL-Klassifikation: D80, H55