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# Stability of the Steady-State Equilibrium in the Uzawa-Lucas Model: A Simple Proof*

By Lutz Arnold**

## 1. Introduction

Uzawa (1965) and Lucas (1988) showed how human-capital accumulation can endogenously generate persistent productivity growth provided that the productivity of education is sufficiently large relative to individuals' time preference. The Uzawa-Lucas model has become a standard tool in the analysis of long-run growth.[1] Nevertheless, the dynamics of the model are not yet generally well-understood. While Uzawa studied the complete dynamics of the model with a linear utility function, Lucas used CES preferences and focused on the steady-state.[2] Caballé and Santos (1993) proved in an important paper that in the absence of externalities the steady state of the Lucas model is globally stable. Their proof is quite complex, however. It makes use of unpublished mathematical results about the concavity and homogeneity properties of the value function for dynamic optimization problems (Santos, 1990).[3] The present paper shows that it is possible to give a very simple proof of the Caballé-Santos stability proposition using the well-known concept of the factor-price frontier. This approach makes the important proof accessible to a much broader audience.

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1 For example, it has been used to investigate the effects of factor taxes by Lucas (1990).

2 Lucas (1988) added externalities from human capital. The model with externalities generates interesting dynamics. In particular, one finds that the steady-state equilibrium may be non-unique and that the equilibrium path may be indeterminate when the steady-state equilibrium is uniquely determined (Xie, 1994, Benhabib and Perli, 1994).

3 Moreover, there is a problem with the convergence notion employed in Caballé and Santos (1993). Caballé and Santos show that, given any variable $x$ in the model, $xe^{-g_x^* t}$ converges, where $g_x^*$ denotes the steady-state growth rate of $x$. As pointed out by Koch (1995), this does not imply convergence of $g_x$ (in what follows, the growth rate of a variable $x$ is denoted as $g_x$). An anonymous referee has pointed out a counter-example: let $Z(t) = (1 + e^{-t} \sin e^t)e^{\gamma t}$ $(\gamma > 0)$; then $Z(t)e^{-\gamma t} = 1 + e^{-t} \sin e^t$ converges but $g_Z$ $(\to \gamma + \cos e^t)$ does not. Such problems do not arise in the present paper.

Section 2 describes the Uzawa-Lucas model of economic growth. Its steady state is analyzed in section 3. Our simple stability proof is found in section 4. Section 5 desribes the transitional dynamics,[4] Section 6 concludes.

## 2. Model

Following Caballé and Santos (1993), we focus exclusively on the Uzawa-Lucas model *without externalities*.[5] Consider a closed economy populated by a continuum $[0,1]$ of identical, infinitely-lived Barro-type families; the size of each family, $N$, equals the total population size, it grows at an exogenous and constant rate: $g_N = n \geq 0$.[6] The economy's output, $Y$, of a homogeneous product equals $F(K, uNh)$, where $K$ denotes physical capital, $u$ is the fraction of the individuals' time devoted to current production, $h$ is human capital per capita, and $F$ is a constant-returns-to-scale production function satisfying all the standard Inada concavity, differentiability, and derivative conditions (the time argument is suppressed if convenient). Output can be consumed or invested: $Y = C + \dot{K}$, where $C$ denotes aggregate consumption (we neglect depreciation for simplicity). The final-goods market is perfectly competitive. Thus, choosing the final good as the numeraire, both the interest rate, $r = f'(k_Y)$, and the wage rate for human capital, $w = f(k_Y) - k_Y f'(k_Y)$, are functions of the capital-human capital ratio *in production*, $k_Y \equiv K/(uNh)$, alone ($f(\cdot) \equiv F(\cdot, 1)$ denotes the production function in its factor intensive form). Moreover, eliminating $k_Y$ from the expressions for the factor rewards yields the factor-price frontier (FPF) $r = r(w)$. The FPF is a continuous hyperbola with $\lim_{w\to 0} r(w) = \infty$ and $\lim_{w\to\infty} r(w) = 0$. There are no pure profits: $Y = rK + wuNh$.

The families' preferences over per-capita, consumption, $c \equiv C/N$, are given by the intertemporal utility function (IUF) $\int_0^\infty e^{-\rho t}[c(t)^{1-\sigma} - 1]/(1-\sigma)dt$, where $\rho > 0$ is the discount rate for future consumption, and $1/\sigma$ is the constant elasticity of substitution between current and future consumption. Consumers have to make two decisions. First, they allocate their human capital to production and education. By devoting a fraction $1 - u$ of their non-

[4] Barro and Sala-i-Martin (1995, section 5) discuss the transitional dynamics of the model using a Cobb-Douglas production function. The present paper generalizes their argument to neoclassical production functions and adds a simple proof of the Caballé-Santos stability proposition.

[5] Another key feature of the Uzawa-Lucas model is the assumption that new human-capital production uses human capital alone. Note that, following Rebelo (1991), some authors assume that human capital production also employs physical capital.

[6] The analysis below becomes somewhat simpler by setting $n = 0$ and choosing units so that $N = 1$.

leisure time to education, they forgo wage income $w(1-u)Nh$, but accumulate additional human capital according to

$$\dot{h} = \phi(1-u)h, \quad \phi > 0. \tag{1}$$

Note that we have implicitly assumed what Mulligan and Sala-i-Martin (1993, p. 743) call constant point-in-time returns in both sectors of production. In this respect, the work of Caball'e and Santos (1993) is more general than ours. However, both Uzawa (1965) and Lucas (1988) adopted constant point-in-time returns in their seminal papers. These are also assumed in Barro and Sala-i-Martin (1995). Second, families allocate their income, $rK + wuNh$, to consumption and savings. Unspent income is used to buy assets. The only available asset is a claim to physical capital. Hence, asset holdings equal the capital stock, $K$, and

$$\dot{K} = rK + wuNh - Nc. \tag{2}$$

## 3. Steady-State Equilibrium

Consumers maximize their IUF subject to the constraints (1) and (2) and the initial conditions $K(0) = K_0$ and $h(0) = h_0$, where $K_0$ and $h_0$ are historically given. The current-value Hamiltonian for this problem is

$$\mathcal{H} = \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda(rK + wuNh - Nc) + \mu\phi(1-u)h,$$

where $\lambda$ and $\mu$ are the multiplier functions for (equilibrium marginal valuations of) $K$ and $h$, respectively. The necessary optimality conditions are[7]

$$\frac{\partial \mathcal{H}}{\partial c} = c^{-\sigma} - \lambda N = 0 \tag{3}$$

$$\frac{\partial \mathcal{H}}{\partial u} = h(\lambda w N - \mu\phi) = 0 \tag{4}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial \mathcal{H}}{\partial K} = \lambda(\rho - r) \tag{5}$$

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[7] The same set of equations is obtained in the optimal-growth problem, in which (2) is replaced by $\dot{K} = F(K, uNh) - Nc$. In this case $w$ and $r$ are abbreviations for the partial derivatives of the production function, not factor prices. The relationship given by the FPF remains valid, however. Thus, the equilibrium path is efficient. This is the kind of efficiency analysis pursued, for example, in Sala-i-Martin (1990a,b). Note that Sala-i-Martin (1990b) does not carry out such an analysis for the Uzawa-Lucas model. He concentrates on the case of household production, in which the equilibrium path coincides with the optimal trajectory by definition.

(6) $$\dot{\mu} = \rho\mu - \frac{\partial \mathcal{H}}{\partial h} = \mu\left[\rho - \frac{\lambda}{\mu} wuN - \phi(1-u)\right]$$

(7) $$0 = \lim_{t\to\infty} e^{-\rho t}\lambda K$$

(8) $$0 = \lim_{t\to\infty} e^{-\rho t}\mu h.$$

Since both constraints ((1) and (2)) as well as the IUF are concave functions, conditions (3)-(8) are sufficient for a maximum. Intuitively, (3) and (4) require that final output and human capital yield the same marginal benefits in their alternative uses. Equation (5) states that the marginal valuation of capital depreciates at a rate equal to the net interest rate, $r - \rho$. The interpretation of (6) is similar. Finally, the transversality conditions (TCs) (7) and (8) rule out explosive paths.

In this section we focus on the steady-state solution of the above system, i.e. we neglect the initial conditions and investigate whether there is a solution to the system in which all variables grow at constant rates. From (4)-(6), $g_w = r - n - \phi$. Thus, the interest rate $r$ is constant in a steady state. Given the FPF, the wage rate $w$ must then be constant, too, i.e. $g_w = 0$. Let the steady-state value of any variable $x$ be denoted by $x^*$. Then $r^* = \phi + n$, $k_Y^* = (f')^{-1}(\phi + n)$, and $w^* = f(k_Y^*) - k_Y^* f'(k_Y^*)$ are uniquely determined. Since $g_\lambda^* = \rho - r^* = \rho - n - \phi$, it follows that $g_c^* = -(g_\lambda^* + n)/\sigma = (\phi - \rho)/\sigma \equiv g^*$ and $g_C^* = g^* + n$. We assume throughout that $\phi > \rho$, so that $g^* > 0$. Note that convergence of the utility integral requires $(1-\sigma)g^* < \rho$, that is $(1-\sigma)\phi < \rho$ (a sufficient condition is $\sigma \geq 1$). Note also that this is equivalent to $g^* < \phi$. Furthermore, $\dot{K} = F(K, uNh) - C$ implies

(9) $$g_K = AP - \chi,$$

where $AP = AP(k_Y) \equiv f(k_Y)/k_Y = F(K, uNh)/K$ denotes the average productivity of capital and $\chi \equiv C/K$ is the consumption-capital ratio. In the steady state $AP^* \equiv AP(k_Y^*)$ is constant. Hence, $\chi$ must be constant, too, so that $g_K^* = g^* + n$ and $\chi^* = AP^* - g^* - n$. $\chi^* > 0$ follows from $AP^* > f'(k_Y^*) = r^* = \phi + n > g^* + n$ (where use has been made of the concavity of $f$ and of $\phi > g^*$). Finally,

(10) $$g_h = \phi(1-u) = \phi\left(1 - \frac{k}{k_Y}\right),$$

where $k \equiv K/(Nh)$ is the overall capital intensity. Constancy of $g_h$ requires $g_k^* = g_{k_Y}^* = 0$. Thus, $g_h^* = g^*$. The equilibrium capital intensity equals $k^* = (1 - g^*/\phi)k_Y^*$.

To sum up: the factor prices, $w^*$ and $r^*$, and the capital intensities, $k_Y^*$ and $k^*$, are uniquely determined and constant in the steady state. Per capita consumption, $c$, human capital per capita, $h$, physical capital per capita, $K/N$, and (because of constant returns to scale) output per capita, $Y/N$, all grow at the constant, positive rate $g^* \equiv (\phi - \rho)/\sigma$. Consumption, $C$, capital, $K$, and output, $Y$, grow at rate $g^* + n$.

## 4. Stability

In this section we show that the economy converges to the steady-state equilibrium described above. The key to the simple proof given here is the observation that the transitional behaviour of the wage rate, $w$, can be analyzed independently of all other variables.[8] Equations (4)-(6) and the FPF yield a differential equation in the wage rate alone:

$$g_w = r(w) - \phi - n.$$

It has already been argued that there is a single stationary solution $w^*$ to this differential equation. But as $r(w)$ is monotonically decreasing, $g_w > 0$ for $w < w^*$ and $g_w < 0$ for $w > w^*$ (see Figure 1). Thus, the wage rate approaches its stationary value $w^*$ in the long run. It follows immediately that $k_Y$, $r$, $AP$, $g_\lambda$, $g_c$, and $g_C$ converge to their steady-state levels. It remains to be shown that $\chi$ and $k$ converge to $\chi^*$ and $k^*$, respectively. Convergence of $\chi$ implies $g_K \to g^* + n$ (and $g_{K/N} \to g^*$); convergence of $k$ implies $g_h = \phi(1 - k/k_Y) \to g^*$ (and $g_{Y/N} \to g^*$, $g_Y \to g^* + n$).

Consider first the consumption-capital ratio, $\chi$. By definition, $g_\chi = g_C - g_K = g_c + n - g_K$. Using (9) and $AP^* - g^* - n - \chi^* = 0$, one obtains

$$g_\chi = g_c + n - AP + \chi = (g_c - g^*) - (AP - AP^*) + (\chi - \chi^*). \tag{11}$$

[8] Barro and Sala-i-Martin (1995), in analyzing the transitional behaviour of the model, do not focus explicitly on the wage rate but on the average product of capital in production. Since they use a Cobb-Douglas production function, this average product is proportional to the marginal product, i.e. the interest rate. But given the FPF, the interest rate is a function of the wage rate. Thus, our approach is essentially a generalization of theirs.

Recently, Bond et al. (1996) and Mino (1996) have presented stability analyses for the generalized model with capital as an input in education. It turns out that this model is accessible to global-stability analysis only under the unrealistic assumption that education is more capital intensive than production (in this case, the relative factor price immediately adopts its steady-state value, and the system becomes effectively two-dimensional). The more realistic case with human-capital intensive education (put in its extreme form in the present paper by assuming away physical capital in the human-capital production function) is subject to local stability analysis only in Bond et al. (1996) and Mino (1996).

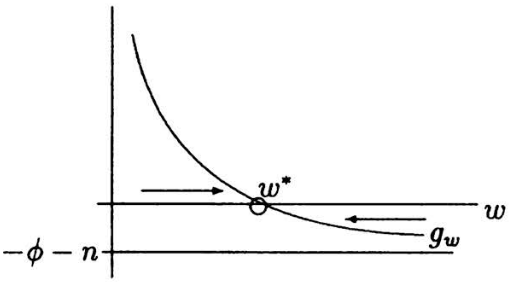


Figure 1: Dynamics of the wage rate

Convergence of $g_c$ and $AP$ implies that the first two terms in parentheses vanish in the long run. This leaves us with an unstable equation in $\chi$: $g_\chi = \chi - \chi^*$. We want to show that $\chi \to \chi^*$ and $g_\chi \to 0$. In this case $g_K \to g^* + n$. Two other situations are conceivable. First, suppose $\chi > \chi^*$. This implies $g_\chi > 0$. It follows that $\chi$ moves away from its steady-state level, and $g_\chi \to \infty$. This implies $g_K = g^* + n - g_\chi \to -\infty$: the capital stock becomes zero in finite time. But since capital is essential in production, $C = 0$ thereafter. This violates the consumers' optimality condition (3). If, on the other hand, $\chi < \chi^*$, then $g_\chi < 0$ and $\chi \to 0$. This implies $g_K \to AP^*$. Since $g_\lambda^* = \rho - n - \phi$, $e^{-\rho t}\lambda K$ grows like $e^{(AP^* - n - \phi)t}$. But as $AP^* > n + \phi$, TC (7) will be violated. Intuitively, $\chi \to 0$ indicates over-accumulation of physical capital. Hence, divergent paths are ruled out as equilibrium candidates, it follows that $\chi$ approaches $\chi^*$ and $g_K$ approaches $g^* + n$.

Next, consider the evolution of the overall capital intensity, $k$. By definition, $g_k = g_K - g_h - n$ or, using equation (10), $g^* = \phi(1 - k^*/k_Y^*)$, and $\chi^* + g^* + n - AP^* = 0$,

$$(12) \qquad g_k = AP - \chi - \phi\left(1 - \frac{k}{k_Y}\right) - n = (AP - AP^*) - (\chi - \chi^*) + \phi\left(\frac{k}{k_Y} - \frac{k^*}{k_Y^*}\right).$$

Convergence of $k_Y$ and $\chi$ entails $g_k = \phi(k - k^*)/k_Y^*$ in the long run. Suppose $k < k^*$. Then, $g_k < 0$, $k \to 0$, and $g_h = \phi(1 - k/k_Y^*) \to \phi$. Thus, $e^{-\rho t}\mu h$ is constant, thereby violating TC (8). Individuals devote their whole non-leisure time to education without using their growing human capital to manufacture output; human capital is over-accumulated. Turning to the case $k > k^*$, for the sake of simplicity, assume that $1 - u < 0$ is possible temporarily at the expense of a decline in human capital ($g_h = \phi(1 - u) < 0$). $k > k^*$ yields $k \to \infty$ and $g_h = \phi(1 - k/k_Y^*) \to -\infty$: human capital becomes zero in finite time, bringing about zero production thereafter, thus violating (3). This proves $k \to k^*$ and, consequently, $g_h \to g^*$.[9]

## 5. Transitional dynamics

The stability result can be illustrated by drawing phase diagrams. In order to avoid case distinctions, assume that the production function exhibits an elasticity of substitution, $\theta$, (not necessarily constant) that is not smaller than unity everywhere. Since

$$\frac{d}{dk_Y}\left[AP(k_Y) - f'(k_Y)\right] = f''(k_Y)\left[\frac{AP(k_Y)}{f'(k_Y)}\theta - 1\right]$$

is negative for $\theta \geq 1$, we notice that the difference between average and marginal productivity decreases as $k_Y$ rises. Consider the phase diagram in $(k_Y, \chi)$ space in Figure 2. Since $k_Y$ converges monotonically, $\dot{k}_Y = 0$ for $k_Y = k_Y^*$, $\dot{k}_Y > 0$ to the left and $\dot{k}_Y < 0$ to the right. From (11), $g_\chi = 0$ for

$$\chi = AP - g_c - n = \frac{\sigma \cdot AP - f' + \rho}{\sigma} - n,$$

which is decreasing in $k_Y$ due to the assumption $\theta \geq 1$ and converges to $\rho/\sigma - n$ as $k_Y \to \infty$. $\chi$ increases (decreases) above (below) the stationary locus. As shown above, there is a unique steady-state equilibrium. Apparently, the steady state is a saddle point, and divergent paths yield either $\chi, g_\chi \to \infty$ or $\chi \to 0$. As shown above, this violates the consumers' optimality conditions. Note that the vertical difference between $AP$ and the $g_\chi = 0$ locus equals $(f' - \rho)/\sigma - n$, which goes to infinity as $k_Y \to 0$. Since the saddle path lies below the $g_\chi = 0$ locus for small values of $k$, this implies $AP - \chi \to \infty$ as $k_Y \to 0$.

Figure 3 depicts a phase diagram in $(k_Y, k)$ space. As in Figure 2, the $\dot{k}_Y = 0$ locus is vertical and stable. From (12), $g_k = 0$ if

$$k = \frac{k_Y}{\phi}(\chi + n + \phi - AP) \equiv \psi.$$

$g_k > 0$ ($< 0$) above (below) $\psi$. The economy follows its saddle path in $(k_Y, \chi)$ space. Hence, from Figure 2, $\chi$ is a function of $k_Y$ and, consequently, $\psi$ is a function of $k_Y$ alone. $\psi$ is negative for small values of $k_Y$ because $AP - \chi \to \infty$ as $k_Y \to 0$. It is positive for $k_Y = k_Y^*$ because $k^* > 0$. Finally, it

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[9] With a non-negativity constraint imposed on $1 - u$, human capital cannot decrease. $h$ would remain constant instead once the constraint $u \leq 1$ becomes binding and the economy would converge to a Ramsey-type steady state with stagnation. We conjecture that this policy would not be optimal either. Artificially, it could be ruled out by postulating a preference for some positive long-run growth. Note also that, similarly, we have not imposed a non-negativity on investment, $\dot{K}$. Thus, we have implicitly assumed that capital can be eaten.

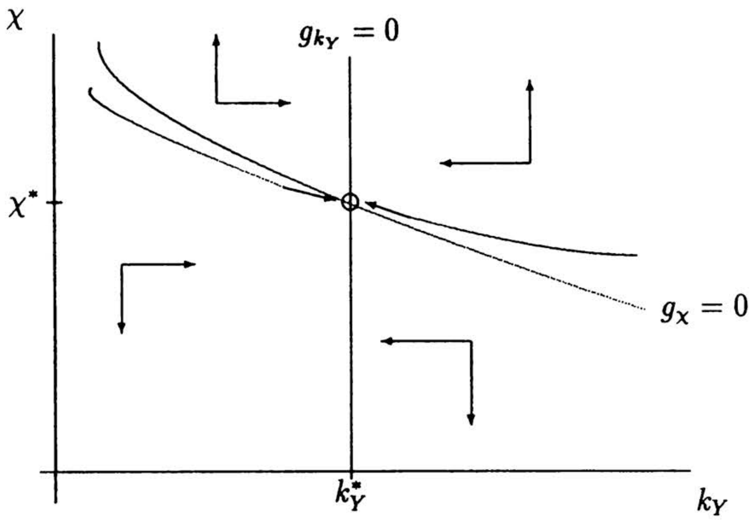


Figure 2: Dynamics of the consumption-capital ratio

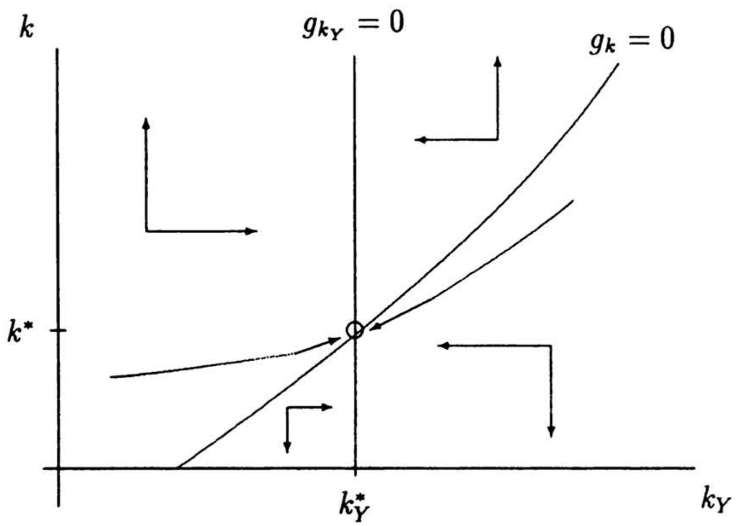


Figure 3: Dynamics of the overall capital intensity

approaches $k_Y[1+\rho/(\sigma\phi)]$ as $k_Y \to \infty$ because $AP$ vanishes and $\chi + n$ goes to $\rho/\sigma$. We assume that $\psi$ is increasing where it is positive, as depicted in Figure 3. Then it is seen from Figure 3 that the equilibrium $(k_Y^*, k^*)$ is a saddle point which is approached monotonically. The historically given initial capital intensity determines the starting point on the saddle path in Figure 3. The implied value of $k_Y$ pins down the initial value of $\chi$ via Figure 2.

It should be noted that, according to Figure 2, $k_Y$ has to be sufficiently small for $g_K$ $(= AP - \chi)$ to be positive. Similarly, $1 - u \geq 0$ requires $k_Y > k$, which may be violated for large values of $k$. These situations will not arise in equilibrium if the initial value of $k$ is sufficiently close to $k^*$. Barro and Sala-i-Martin (1995, p. 196) estimate in their Cobb-Douglas framework that the boundary conditions do not bind for values of $k$ ranging between 5% and 500% of its steady-state value. Another remark is in order. We perceive that economies with equal initial capital intensities follow the same trajectory in $(k_Y, k)$ space irrespective of the absolute values of the initial endowments $K_0$ and $h_0$. This implies that proportional differences in factor endowments are preserved in the growth process (as in the simple „AK model“).[10] Although growth rates converge, richer countries remain proportionally richer. Finally, Figure 3 sheds some light on the workings of the model: if the overall capital intensity is small, then $k_Y$ is small too, the interest rate is high, and the wage rate is low, as in the neoclassical model.

## 6. Conclusions

We have shown that stability of the steady-state equilibrium in the Uzawa-Lucas model can be proved in a much simpler way than it has been done in the sophisticated work of Caballé and Santos (1993).[11] The key element of the simplified proof is to first examine the dynamics of the competitive wage rate. It has been shown (i) that a unique steady-state equilibrium exists, (ii) that the wage rate approaches its steady-state value, and (iii) that convergence of the wage rate implies convergence of all other variables.

10 This means that the steady state is unique in terms of $k_Y$ and $k$ but not in terms of $K$ and $h$. In other words: there is a ray of steady states with a uniform capital intensity.

11 And that, at the same time, the problems mentioned in footnote 3 can be avoided.

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## Zusammenfassung

Das Uzawa-Lucas-Modell analysiert endogenes Wirtschaftswachstum durch Humankapitalakkumulation. Meist wird nur eine Steady-state-Analyse durchgeführt, bekannte Stabilitätsbeweise sind schwierig. Der vorliegende Artikel liefert einen sehr einfachen Beweis globaler Stabilität im Uzawa-Lucas-Modell.

## Abstract

The Uzawa-Lucas model studies human-capital accumulation as an engine of growth. While the focus is mostly on the steady state, available stability proofs are difficult. The present paper offers a simple proof of the global stability of the steady state.



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