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Product Differentiation and the Intensity of Price Competition*

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1. Introduction

The Hotelling model of spatial competition is a convenient tool for modelling product differentiation. It can literally describe the choice of locations by firms when customers are distributed along a one-dimensional geographic entity, like a road or a river. But more frequently, it is used to model the choice of product characteristics by firms under the simplifying hypothesis that tastes of consumers differ only in one dimension.[1]

In Hotelling's original article (1929) and in much of the subsequent work consumers are supposed to be distributed uniformly along the line. However, as Neven (1986) points out, increasing densities toward the center of the distribution have more appeal in many situations. For example, most cities exhibit areas of greater agglomeration at the center. Likewise, if the distribution describes consumers' tastes for certain product characteristics, it seems reasonable to suppose that more extreme tastes are less frequent than 'average' tastes.

In this note we show that with such reasonable consumer distributions one encounters problems with the existence of equilibrium, which are different from the ones described earlier in the literature. D'Aspremont et al. (1979) were first to point out that there was a flaw in Hotelling's original argument. Namely, the existence of a pure strategy equilibrium in the price subgame cannot be assured for all location choices of the two firms if consumers are distributed uniformly and transportation costs are linear. They also demonstrate that the problem disappears if transportation costs are quadratic. More generally, Caplin and Nalebuff (1991) show that a pure strategy equilibrium in the price game exists and is unique for a broad

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1 Generalizations to higher dimensional spaces are possible to some extent (see e.g. Tabuchi, 1994).

range of consumer utility functions (including the case of quadratic transportation costs) if the consumer density function $f(x)$ is log-concave.

We assume a log-concave density function and quadratic transportation costs to make sure that there will be no problems with the existence of equilibrium in the price subgame. Nevertheless, a pure strategy equilibrium in the two-stage game may still fail to exist. This observation was already made by Tabuchi and Thisse (1995) and Goeree and Ramer (1994), in particular for the case of a triangular density of consumers. Tabuchi and Thisse (1995) conjecture that the non-existence also holds for densities which are differentiable but very steep at the midpoint. This note is an extension of their work and we prove that their conjecture was correct. We concentrate on completely symmetric games, that is, games with symmetric consumer densities and symmetric cost structures for both firms. We explain why a symmetric equilibrium need not exist by decomposing the consequences of a small change in location by one firm into three effects. We identify a *price effect*, a *quantity effect* and a *competition intensity effect*. What turns out to be essential for the necessary condition for an equilibrium to hold is that the *competition intensity effect* must not be too large. The latter effect measures how strongly the intensity of price competition decreases when a firm moves away from its competitor.

Usually, when one considers a symmetric game, the conjecture is that *if an equilibrium exists at all*, then a symmetric equilibrium can be found, too. However, in Section III we give an example of a Hotelling game with a symmetric density in which no symmetric equilibrium exists, despite the fact that there are two asymmetric equilibria. In the asymmetric equilibria one firm locates outside the support of the customer distribution, which provides just another example in which Hotelling's 'principle of minimum differentiation' fails to hold.

2. The Model

We consider the standard Hotelling location-price game with quadratic transportation costs and unrestricted locations. In a first step, both firms simultaneously choose their locations on the real line, which we denote by a and b, respectively, (with $a \leq b$). In a second step, given these locations and the known distribution of consumers $F(x)$, firms simultaneously choose their prices. Consumers buy one unit of the good from the supplier with the lower price including transportation cost. Thus, the total price a consumer located at $x \in R$ has to pay is $p_1 + (x-a)^2$ when buying from firm 1 (located at a) or $p_2 + (x-b)^2$ when buying from firm 2 (located at b).

For given locations and prices the position ξ of the consumer who is indifferent between buying from firm 1 or firm 2 is given by

$$\xi(a,b,p_1,p_2) = \frac{1}{2}\left(\frac{p_2 - p_1}{b-a} + b + a\right) . \tag{1}$$

Obviously, all consumers to the left of ξ will buy from firm 1 and all consumers to the right will buy from firm 2. Assuming, for simplicity, that production costs are zero, profits are then given by $\Pi_1(a,b,p_1,p_2) = p_1 F(\xi)$ and $\Pi_2(a,b,p_1,p_2) = p_2\big(1 - F(\xi)\big)$.

Since we want to consider symmetric game situations, we assume that the consumer density $f(x)$ is symmetric and uni-modal. In order to guarantee the existence of a unique solution in the price-subgame we will additionally assume that $f(x)$ is log-concave.

Assumption 1: The consumer density $f(x) = F'(x)$, with convex support $B \subseteq R$, is log-concave, twice differentiable, and symmetric with a mode at $x = 0$.

Applying the Brékopa-Borell Theorem, Caplin and Nalebuff (1991) have shown that log-concavity of the consumer density is sufficient to guarantee existence and uniqueness of an equilibrium in the price game.

Lemma 1: Given Assumption 1 there exists a unique Nash equilibrium of the price game given by the solutions $p_1^*(a,b)$ and $p_2^*(a,b)$ to

$$p_1(a,b) = 2(b-a)\frac{F\big(\xi(a,b,p_1,p_2)\big)}{f\big(\xi(a,b,p_1,p_2)\big)}$$

and

$$p_2(a,b) = 2(b-a)\frac{1 - F\big(\xi(a,b,p_1,p_2)\big)}{f\big(\xi(a,b,p_1,p_2)\big)} .$$

Proof: The formulas for $p_1(a,b)$ and $p_2(a,b)$ follow directly from the first order conditions of profit maximization. The existence of a Nash equilibrium follows from Theorem 2 in Caplin and Nalebuff (1991) and the uniqueness from Proposition 7 in Caplin and Nalebuff (1991).

Using the result of Lemma 1 together with (1) we can implicitly derive the location ξ^* of the indifferent consumer at the equilibrium prices

$$\xi^*(a,b) = \frac{1}{2}(a+b) + \frac{1 - 2F\big(\xi^*(a,b)\big)}{f\big(\xi^*(a,b)\big)} . \tag{2}$$

Note that in a symmetric equilibrium (in pure strategies) $b = a$ and therefore $\xi^* = 0$.

Finally, the reduced form profit functions are given by

$$\Pi_1(a,b) = 2(b-a)\frac{F(\xi^*)^2}{f(\xi^*)} \tag{3}$$

and

$$\Pi_2(a,b) = 2(b-a)\frac{\left(1-F(\xi^*)\right)^2}{f(\xi^*)}\,. \tag{4}$$

The first order condition for profit maximization is

$$\frac{\partial \Pi_1}{\partial a} = -2\frac{F^2}{f} + 2(b-a)\left[2F - \frac{F^2}{f^2}f'\right]\xi_a = 0,$$

where ξ_a denotes the partial derivative of $\xi^*(a,b)$ with respect to a. Straightforward differentiation of (2) yields

$$\xi_a = \frac{f^2}{6f^2 - 2(2F-1)f'}\,,$$

which implies, using symmetry, that $\xi_a|_{-a=b} = 1/6$ because $F(0) = 1/2$. The same holds for ξ_b. Hence,

$$-\mathrm{a}^* = \mathrm{b}^* = \frac{3}{4f(0)}$$

is the only candidate for a symmetric equilibrium in pure strategies.

What matters for the existence of a symmetric equilibrium are the consequences a change in location by one firm has on prices and quantities. We can decompose those effects by defining the following elasticities. Let

$\eta_{p_1^*,a} \equiv \dfrac{\partial p_1^*(a,b)}{\partial a}\dfrac{a}{p_1^*(a,b)}$ denote the elasticity of the equilibrium price with respect to the location of firm 1,

$\eta_{F,a} \equiv \dfrac{a}{F}\dfrac{\partial F(\xi^*)}{\partial a}\Big|_{p_1^*=p_2^*=\text{const.}}$ denote the elasticity of quantity sold with respect to location for *given* prices,

$\eta_{\left(\frac{\partial p_1^*}{\partial a}\right),a} \equiv \dfrac{\partial^2 p_1^*}{\partial a^2}\dfrac{a}{\frac{\partial p_1^*}{\partial a}}$ denote the elasticity of the reaction in price to location as location is varied.

The first elasticity indicates the direct *price effect* of a change in location, the second the direct *quantity effect* and the last one the indirect *competition intensity effect*.

Proposition: A symmetric equilibrium in pure strategies exists only if $\eta_{p_1^*,a} + \eta_{F,a} + \eta_{\left(\frac{\partial p_1^*}{\partial a}\right),a} \leq 0.$

Proof: We prove the Proposition by demonstrating that the symmetric candidate solution can be a maximum only when it satisfies the above condition.

Simple calculations show that for all densities satisfying Assumption 1 $\eta_{p_1^*,a} = 1/4$ and $\eta_{F,a} = -3/4$ at the symmetric candidate solution. The necessary condition for the existence of a symmetric equilibrium depends, therefore, entirely on the size of the competition intensity effect. It is satisfied if and only if

$$\eta_{\left(\frac{\partial p_1^*}{\partial a}\right),a} \leq \frac{1}{2} \, .$$

To check whether the candidate solution represents in fact a (local) maximum, we calculate the second order conditions.

$$\frac{\partial^2 \Pi_1}{\partial a^2} = -4\xi_a\left(2F - \frac{F^2}{f^2}f'\right) + 2(b-a)\left[\left(2F - \frac{F^2}{f^2}f'\right)\xi_{aa} + \xi_a^2\left(2f - 2Ff'\frac{f^2 - Ff'}{f^3} - \frac{F^2}{f^2}f''\right)\right]$$

Using the fact that $f'(0) = 0$ and $F(0) = 1/2$, further differentiation of (2) yields $\xi_{aa}|_{-a=b} = 0$. When we substitute the above calculated values for a^*, b^*, $F(0)$, $f'(0)$ and $\xi_{aa}|_{-a=b}$, we find that

$$\frac{\partial^2 \Pi_i}{\partial a^2}\,|_{a^*} \leq 0 \Leftrightarrow 24f(0)^3 \geq -f''(0)\,. \tag{5}$$

Consider now $\eta_{\left(\frac{\partial p_1^*}{\partial a}\right),a}$. Since

$$\frac{\partial p_1}{\partial a} = -2\frac{F}{f} + 2(b-a)\xi_a\left(1 - \frac{Ff'}{f^2}\right)$$

and

$$\frac{\partial^2 p_1}{\partial a^2} = -4\xi_a\left(1 - \frac{Ff'}{f^2}\right) + 2(b-a)\left[\xi_{aa} - f'\left(\frac{f^2\xi_a^2 - 2Ff'\xi_a^2 + \xi_{aa}fF}{f^3}\right) - \frac{f''\xi_a^2 F}{f^2}\right],$$

one finds that at the symmetric solution

$$24f(0)^3 \geq -f''(0) \Leftrightarrow \eta_{\left(\frac{\partial p_1^*}{\partial a}\right),a} \leq \frac{1}{2}\,.$$

Generally speaking there is always a trade-off in the choice of location. One the one hand, a firm has an incentive to locate as close as possible to the center since the bulk of customers is located there. On the other hand, as firms get closer to each other, competition in the price subgame becomes fiercer. The steeper the consumer density is at the center (which is the location of the indifferent consumer in the symmetric candidate solution), the higher will be the intensity of competition for customers located there.

Our condition states that for a symmetric equilibrium to exist the increase in the price that can be charged is not too steep as a firm moves away from the center. In other words, the intensity of competition must not decrease too much when moving away from the center. The equivalent condition $24f(0)^3 \geq -f''(0)$ demands that the consumer density must not be too steep at the center in the sense that $f''(0)$ must not be too large in absolute value relative to $f(0)$.

If the condition is violated, we get a profit minimum instead of a maximum. This implies that when firm 2 locates at the symmetric candidate solution b*, firm 1 has an incentive to deviate from a* in *both* directions. A deviation to the center is profitable because so many new customers can be gained. And a deviation away from the center is profitable because the competition intensity is reduced so drastically that the loss of customers is more than offset by the possibility to charge a higher price.

Remark 1: In order to check which symmetric densities $f(x)$ satisfy the condition stated in the Proposition, it is easier to examine the equivalent condition $24f(0)^3 \geq -f''(0)$. As pointed out above, the distribution must not be too steep at the center. In particular, it must be differentiable at the center (see Tabuchi and Thisse, 1995; and Goeree and Ramer, 1994, for this result) as non-differentiability can be seen as the limit case of a differentiable but very steep density. For example, with a triangular distribution a symmetric equilibrium does not exist. On the other hand, it may easily be checked that many common distributions, like the Normal distribution, do satisfy the condition.

Remark 2: Note that the condition given in the Proposition is only necessary but not sufficient for a symmetric equilibrium. As the condition is derived from the second order conditions for profit maximization, it guarantees only a local maximum. To guarantee that a candidate solution represents in fact a global maximum, one has to impose stronger conditions (see Goeree and Ramer (1994) for such a (rather complicated) sufficient condition).

3. An Example

As mentioned above a simple, but surprising, example for the non-existence of a symmetric equilibrium in a completely symmetric Hotelling location game is the case of a triangular consumer density, e.g. $f(x) = 1 - |x|$.[2] Since the location of the indifferent consumer ξ^* is of decisive importance in the analysis and since the density is non-differentiable at exactly this point in the case of symmetric locations, we thought it at first plausible to suspect the non-differentiability to cause the non-existence.[3] However, as the analysis of the last section showed there is a non-trivial necessary condition for the existence of a symmetric equilibrium. Intuitively, the condition requires that the density be not too steep at the peak. We will now give an example of a density that violates this condition.

The following density satisfies all aspects of Assumption 1: it is log-concave, symmetric and twice differentiable everywhere. It was constructed by smoothing the triangular density at the peak by inserting a polynomial for some range at the center of the density.

$$f(x,c) = \begin{cases} n(c)(1+x) & -1 \leq x \leq -\frac{1}{2c} \\ n(c)(1 - \frac{3}{16c} - \frac{3}{2}cx^2 + c^3x^4) & -\frac{1}{2c} \leq x \leq \frac{1}{2c} \\ n(c)(1-x) & \frac{1}{2c} \leq x \leq 1 \end{cases}$$

where $c > 1/2$ is a parameter and $n(c) = \frac{20c^2}{20c^2-1}$ is a normalization factor such that $\int_{-1}^{1} f(x,c)\mathrm{d}x = 1, \forall c > 1/2$. It is straightforward to check that the condition of $24f(0)^3 \geq -f''(0)$ is violated for all $c > 7.42$, which implies that the symmetric candidate solution is a local profit minimum instead of a maximum. Equivalently, it can be checked that the *competition intensity effect* represented by $\eta_{\left(\frac{\partial p_1^*}{\partial a}\right),a}$ becomes larger than 1/2 for all $c > 7.42$. Figure 1 displays the profit function of firm 1 for the case of $c = 20$ if firm 2 is located at the symmetric candidate solution $b^s = 0.7570$. It is apparent that $\Pi_1(a, b^s)$ fails to be quasi-concave.

Due to the lack of quasi-concavity of the profit functions the reaction functions[4] (i.e., the optimal location choice of firm i for a given location of firm j assuming that prices are chosen optimally) are discontinuous (see Figure 2) and therefore no symmetric equilibrium exists.[5] Furthermore, it can

[2] See Goeree and Ramer (1994) or Tabuchi and Thisse (1995) for a proof.

[3] However, Tabuchi and Thisse (1995) conjectured, but did not prove, that the non-existence also occurs with differentiable densities.

[4] Strictly speaking, they are reaction correspondences since at the jump both end-points are best replies.

[5] The reaction functions are *not* piece-wise linear although it might appear so in the graph.

be seen that despite the non-existence of a symmetric equilibrium two asymmetric equilibria exist (where the reaction functions intersect). For the case of c = 20 the asymmetric equilibria are located at a_1^* = -1.5443, b_1^* = 0.3607 and a_2^* = 0.3607, b_2^* -1.5443, respectively. Note that one firm locates outside of the support of the customer distribution.

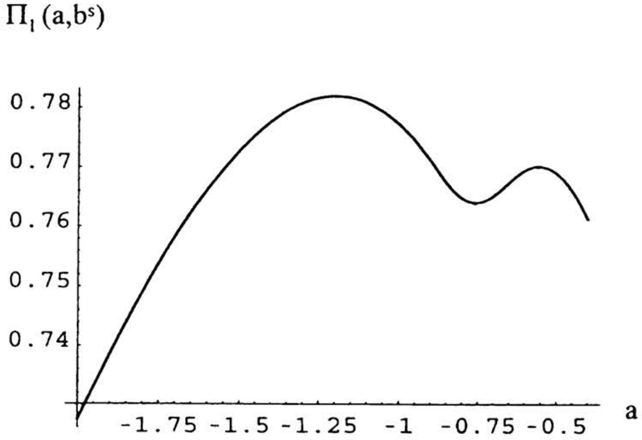

Figure 1: Profit function of firm 1 with b^s = 0.7570 and c = 20

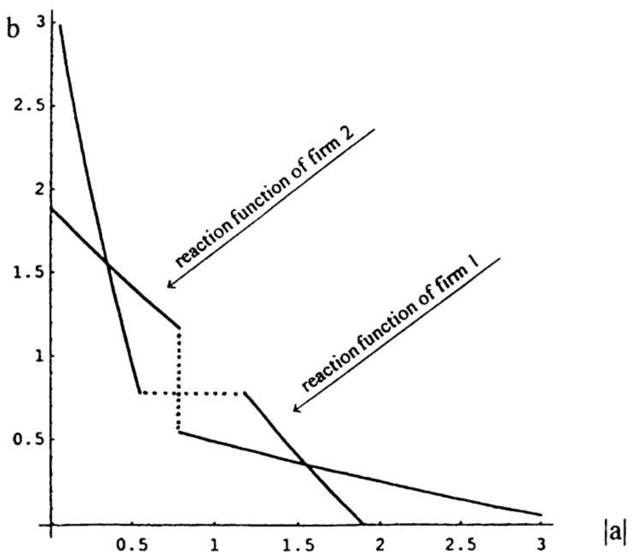

Figure 2: Reaction functions for c = 20

However, even if $c < 7.4224$, the symmetric candidate is not necessarily an equilibrium. In fact, calculations show that for $6.415 < c < 7.4224$ the symmetric candidate is only a local maximum. Only for $c < 6.415$ a symmetric equilibrium exists.

4. Conclusion

This note points out a problem with Hotelling-type product differentiation models. When customers are concentrated near the center of the distribution, which seems to be a realistic assumption in many settings, the two-stage game may fail to have a symmetric pure strategy equilibrium. We relate this existence problem to the "competition intensity effect" and show that a symmetric equilibrium does not exist if the competition intensity decreases too rapidly when a firm moves away from its competitor.

Furthermore, we show that even in situations in which no symmetric equilibria exist, there may still be asymmetric equilibria. These asymmetric equilibria are characterized by strong product differentiation between the firms, as one firm locates even outside the support of the customer distribution.

References

Caplin, A. and B. *Nalebuff,* 1991, Aggregation and Imperfect Competition: On the Existence of Equilibrium, Econometrica, 59, 25 - 59.

D'Aspremont, S., C. *Gabszewizc* and J.-F. *Thisse,* 1979, On Hotelling's Stability in Competition, Econometrica, 47, 1145 - 1150.

Goeree, J. and R. *Ramer,* 1994, Exact Solutions of Location Games, Tinbergen Institute Discussion Paper No. 94 - 70, Amsterdam University.

Hotelling, H. 1929, Stability in Competition, Economic Journal, 39, 41 - 57.

Neven, D., 1986, On Hotelling's Competition with Non-Uniform Customers Distributions, Economics Letters, 21, 121 - 126.

Tabuchi, T., 1994, Two-stage Two-dimensional Spatial Competition between Two Firms, Regional Science and Urban Economics, 24, 207 - 227.

Tabuchi, T. and J.-F. *Thisse,* 1995, Asymmetric Equilibria in Spatial Competition, International Journal of Industrial Organization, 13, 213 - 227.

Zusammenfassung

Wir betrachten ein Produktdifferenzierungsmodell à la Hotelling, bei dem die Konsumenten symmetrisch um das Zentrum der Verteilung konzentriert sind, was in vielen Fällen einer realistische Annahme ist. Wir erklären, warum die Existenz eines symmetrischen Gleichgewichts nicht gesichert ist. Der Grund ist, daß die Wettbe-

werbsintensität u.U. drastisch reduziert wird, wenn sich die Firma von ihrem Wettbewerber entfernt. Trotz der Nichtexistenz von symmetrischen Gleichgewichten gibt es asymmetrische Gleichgewichte, welche jedoch von starker Produktdifferenzierung geprägt sind. In vielen Fällen siedelt sich eine der Firmen außerhalb des Trägers der Konsumentenverteilung an.

Abstract

We consider a Hotelling-type product differentiation model with customers that are concentrated symmetrically around the center of the distribution, which seems to be a realistic assumption in many settings. We explain why a symmetric equilibrium may fail to exist. The reason is that the intensity of competition may decrease drastically when a firm moves away from its competitor. Despite the non-existence of symmetric equilibria there may exist asymmetric equilibria but those are generally characterized by strong product differentiation between the firms, with the possibility that one firm locates outside the support of the customer distribution.

JEL-Klassifikation: L13, R32

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