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Binomial Pricing of Interest Contingent Assets

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The pricing of interest contingent assets (e.g. bond, swaps, bond options, interest rate futures) has become a major topic in the area of financial asset pricing. This is partially due to the immense growth of these instruments, with respect to both market volume and variety of available instruments. A major academic challenge is to develop a unified pricing approach for these instruments, incorporating the stochastic assumptions on the underlying interest rates in a consistent way in order to exclude arbitrage profits. The contingent claim model is the most prominent approach. In this article, the (spot) interest rates are modelled as a binomial process, and a general recursive valuation formula is derived.

1. Motivation

The pricing of interest contingent assets has become a major issue in the theory of finance. From a practical standpoint, the growth and impressive variety of these instruments calls for sophisticated hedging and pricing models. While the present value model is still the basic approach to consistently incorporate the (observed) term structure of interest rates to the pricing of bonds, it does not incorporate the *stochastic assumptions* about the term structure as a major determinant of asset prices[1].

The contingent claim valuation approach provides an elegant analytical framework to derive consistent, i.e. arbitrage-free asset prices with respect to the stochastic process of the yield curve. From a practical standpoint, a major advantage of the contingent claim aproach is *consistency*: Bond ABC, futures on bond ABC, options on the futures on bond ABC, to give an example, are all priced by the same model relying on the same stochastic assumptions on the interest rate process(es). Specifically, „consistence" means that the valuation model prevents arbitrage opportunities with respect to the underlying interest rate process.

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[1] 'Term structure', 'yield curve' or 'future interest rates' is used more or less synonymously in this paper.

The basic insight is that interest contingent assets can be priced like any other contingent claim. Remember that e.g. the value of a stock option can be determined with respect to the stochastic process of the underlying stock S and time to maturity, $C(S, t)$, neglecting the parameters which are supposed to be constant during the life of the option. The contingent claim approach for interest contingent assets is similiar: The price of a claim B can be determined with respect to the stochastic process governing the underlying interest rate i and time to maturity t, $B(i, t)$. Of course, this assumes that the stochastic movement of the yield curve can be expressed by one single state variable. If more factors are empirically relevant, the number of interest rate processes should be increased accordingly.

Examples of arbitrage based contingent claim models for interest dependent assets are e.g. *Brennan / Schwartz* (1977), *Dothan* (1977), *Vasicek* (1978). A benchmark model for a respective general equilibrium model is *Cox / Ingersoll / Ross* (1985). In all these models, the underlying interest rate follows a continuous time stochastic process. *Cox / Ross / Rubinstein* (1979) have shown that a binomial process for the underlying asset considerably simplifies (or at least clarifies) the pricing of contingent claims. In this paper, this approach is applied to the pricing of interest contingent claims: It is assumed that the process of one period interest rates follows a binomial random walk. Thereby, the (empirical) question is not addressed whether this distributional assumption is a reasonable characterization of observed spot intest rates or not. It is a pure methodological paper to derive a simple pricing formula *given* that interest rates can be modeled in this fashion. Similiar, but different, binomial approaches are those by *Rendleman / Bartter* (1980) and *Ho / Lee* (1987) (see section 7 for a comparison).

The rest of the paper is organized as follows. In the next two sections (2 and 3), the basic characteristics of our approach are outlined: a one-factor term-structure model with binomial interest rate movements. In section 4, the respective arbitrage-based „bond" valuation model is derived. It is a simple recursive algorithm to discount future cash flows in the presence of (binomial) interest rate risk without providing arbitrage profits. In section 5, it is shown that the model is fully consistent with the well-known continuous-time valuation model: the binomial valuation model can be easily transformed to the respective partial differential equation which holds in continuous time. In section 6, some simple applications of the binomial valuation model are presented: an arbitrage example with inconsistently priced bonds; the valuation of callable coupon bonds; and the pricing of bond options, forward and futures contracts on bonds. In the final section (7), our approach is compared to existing models in the literature, and some practical problems are addressed.

2. One factor term structure

One of the most important results in financial theory is the insight that expected returns on risky assets are linked through arbitrage conditions (*Ross* 1976). This insight is applied to the pricing of bonds with different maturities, i.e. to derive the term structure of interest rates given a specific stochastic process for the one period interest rates.

The key insight of asset pricing is that the expected excess return on assets, standardized by their respective risk, must be equal across assets in order to avoid riskless arbitrage profits, formally

$$\frac{E(R_i) - R}{\theta_i} = \frac{E(R_j) - R}{\theta_j} \tag{1}$$

where $E(R_i)$ denotes the expected return on asset i and R is the return on a riskless asset. The ratio is of course the market premium for the underlying risk. The main difference between the various pricing models is the modelling of the underlying risk, and as a consequence, the structure of the risk premium and the measurement of the individual security risks θ_i. In the capital asset pricing framework the underlying risk is the variance of the end of period value of the aggregate market portfolio (i.e. the universe of all existing assets), and the relevant risk of individual assets is the covariance between asset returns and the return on the market portfolio. More generally, the underlying risk may be represented by an arbitrary risk factor F. In the simplest case the return on any asset i is linearly related to F by

$$R_i = a_i + b_i F \tag{2}$$

which is called a single factor representation of the security returns. a_i and b_i are bond-specific constants, and F is a random variable with zero expectation and a finite variance. Note that all returns are perfectly correlated in this case; there is no asset specific (idiosyncratic) risk. The covariance between returns is given by $b_i b_j$ Var (F). This structure can be extended to include several factors, $F_1, \ldots, F_n$ or asset specific risk. In this paper a *single* factor representation of the term structure of interest rate is examined. The term structure is derived from the present value of discount bonds with various maturities. Hence, a single factor representation means that bond specific (e.g. default) risk is neglected and that the risk and expected return of different (discount) bonds is exclusively caused by different bond maturities. Therefore equation 2 represents the relevant generating model for bond returns i.e. the term structure of interest rates throughout the paper.

Consider two default free discount bonds with maturities t and T with $t < T$ and returns $R(t)$ and $R(T)$. Returns will be measured over finite time intervals. The return generating process is given by equation (2). The risk factor F does not need to be specified at this stage. Consider a portfolio where a fraction w is invested in bond t and the remainder, $1 - w$, is invested in bond T. Given the return generating process in equation (2), the returns on bond t and T[2] are perfectly correlated, it is possible to combine a long (short) position in t with a short (long) position in T such that the resulting portfolio is completely hedged against F, i.e. the portfolio return is riskless (*Ingersoll* 1987, 167)[3]. It can easily be shown that the respective portfolio weights are determined by

$$(3) \qquad w^* = \frac{1}{1 + b_t/b_T}, \; 1 - w^* = \frac{1}{1 + b_T/b_t}$$

where w^* denotes the fraction of t period bonds and $1 - w^*$ the fraction of T period bonds in the arbitrage portfolio. Obviously the return on this portfolio must be equal to the riskfree interest rate in order to avoid arbitrage, i.e. $w^* R_t + (1 - w^*) R_T = R$. If the portfolio weights (3) are inserted into this expression, then equation

$$(4) \qquad \frac{a_t - R}{b_t} = \frac{a_T - R}{b_T}$$

results. This is just a special case of equation (1) given the bond return generating process imposed by (2). This derivation is equivalent to the Black-Scholes option pricing methodology where a riskless stock-option position is established in order to derive the fundamental partial differential equation of option valuation[4].

3. Binomial Interest Rates

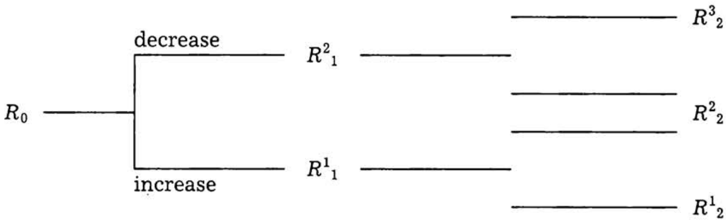
So far no specific distributional assumption has been made for the underlying risk factor F. For the rest of the paper it is assumed that F follows a binomial random walk[5]. If F is the one period (i.e. riskfree) interest rate one period ahead, this implies that the process of one period interest rates is a binomial random walk, graphically

[2] Throughout the paper the bond maturing in t is called "bond t" and the bond maturing in T is called "bond T".

[3] If bonds exhibit specific (idiosyncratic or stochastic residual) risk, this proposition holds asymptotically for sufficiently "large", well diversified portfolios.

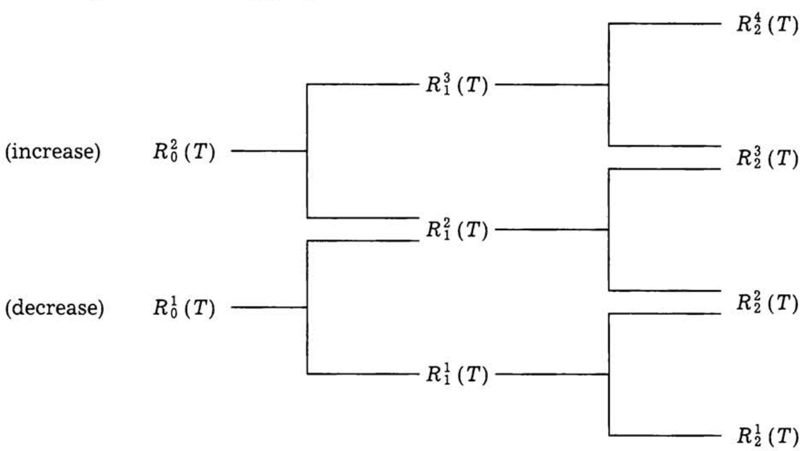
[4] See *Merton* (1977) for a general exposition of contingent claim valuation in the Black-Scholes framework.

[5] See *Cox / Rubinstein* (1985) for a discussion of such processes.



R_t^j is the simple interest rate on an investment from t to $t+1$ in state j. Note that interest rates are set *at the beginning* of each period so that the "first" return in the three, R_0, ist nonstochastic. The probability of a *decrease* of the one period interest rate to R_1^2 is denoted by p and is assumed to be state and time-independent; an increase in the interest rate to R_1^1 occurs with probability $1-p$. R_1 refers to the respective random variable; however, for simplicity the subscript 1 will be dropped subsequently. Also, a "closed tree" is arbitrarily assumed[6].

The binominal bond T returns are defined equivalently. Note that the two possible interest rates at the *end* of the current period, R_1^1 and R_1^2, imply two possible bond prices at the end of the current period, $B_1^1(T)$ and $B_1^2(T)$, and hence, two possible bond returns for the *current* period. They will be denoted by $R_0^1(T)$ and $R_0^2(T)$. Because interest rates and bond prices are negatively correlated, $R_0^2(T)$ corresponds to the falling interest rate scenario, i.e. occurs with probability p and exceeds $R_0^1(T)$. $R_0(T)$ denotes the respective random variable; as in the case of the short rate R_1, the subscript will be dropped for simplicity. The evolution of bond T returns is clarified by the following graph:



[6] This means that $R_0ud = R_0du$, $R_1^2ud = R_1^2du$, and so forth; u and d denote the ratio of two subsequent interest rates and represent the up and down volatility factors of the interest rate movement.

Given the distributional assumptions about F_t and hence R_t it is easy to determine the parameters a_t, b_t and a_T, b_T which are essential to specify the hedging condition (3) and the subsequent arbitrage equation (4). By construction a_T is the conditional expectation of the bond T return $R_0\,(T)$:

$$a_T = E\,[R_0\,(T)] = p\,R_0^2\,(T) + (1 - p)\,R_0^1\,(T) \tag{5}$$

The binomial interpretation of b_T is also straightforward. Note that, from Section 1, b_T is the sensitivity of the return on bond T with respect to the volatility of the underlying factor risk F (the one period interest rate R), $dR\,(T)\,/\,dF = dR\,(T)\,/\,dR$. b_T can alternatively be interpreted as the regression coefficient of two perfectly correlated random variables[7]. Since $b_T = \mathrm{Cov}\,[R\,(T), R]\,/\,\mathrm{Var}\,(R)$ in an „ordinary" regression equation with a stochastic regressor R, the covariance expression can be replaced by the product of standard deviations $\sigma\,[R\,(T)]\,\sigma\,[R]$, and thus the coefficient becomes

$$b_T = \frac{\sigma\,[R\,(T)]\,\sigma\,[R]}{\sigma^2\,[R]} = \frac{\sigma\,[R\,(T)]}{\sigma\,[R]} \tag{6a}$$

which is the ratio of the bond T return volatility with respect to the volatility of the underlying factor (the one period interest rate). With binomial interest rates this corresponds to a ratio in which the volatilities are substituted by differences between the possible returns in state 1 and 2; thus the coefficient is

$$b_T = \frac{R_0^2\,(T) - R_0^1\,(T)}{F_1^1 - F_1^2} = \frac{R_0^2\,(T) - R_0^1\,(T)}{R_1^1 - R_1^2} > 0 \tag{6b}$$

Relations (5) and (6) naturally also apply to bond t. b_T is the binomial equivalent to $dR\,(T)\,/\,dR$ if the short (one period) rate R changes instantaneously. More intuitively it is the binomial equivalent to the traditional "bond volatility" (or "modified duration") of bond T in continuous time. The fact that bond returns are *locally* (i.e. over one period) linear in the underlying risk will be used in Section 4. Inserting equations (5) and (6) into (4) and eliminating $\sigma\,[R]$ yields

$$\frac{E\,[R\,(t)] - R}{\sigma\,[R\,(t)]} = \frac{E\,[R\,(T)] - R}{\sigma\,[R\,(T)]} \tag{7}$$

---

[7] Recall that there is no bond specific risk.

$$
\begin{aligned}
\text{with } E[R(T)] &= p\,R_0^2(T) + (1-p)\,R_0^1(T)\\
\sigma[R(T)] &= R_0^2(T) - R_0^1(T) > 0\\
R &= R_0
\end{aligned}
$$

which is our binomial no-arbitrage condition for bond returns.

4. Binomial Bond Valuation

The previous derivation can be used to derive a bond pricing formula. The fundamental arbitrage condition which follows from the derivation in Sections 1 and 2 is that the risk-adjusted expected excess return, $\{E[R(T)] - R\} / \sigma[R(T)]$, must be equal for all bonds. This ratio shall be denoted by L and called market price of interest rate risk. Using this definition, equation (7) may be written as

$$
\begin{aligned}
(8)\qquad & p\,R_0^2(T) + (1-p)\,R_0^1(T) - L\,[R_0^2(T) - R_0^1(T)]\\
&= (p - L)\,R_0^2(T) + [1-(p-L)]\,R_0^1(T)\\
&= R_0
\end{aligned}
$$

which shows how the statistical probabilities p and hence expected return on the risky bond must be "modified" such that risk averse individuals are willing to hold the risky bond T given the riskfree return R. The modification occurs by subtracting the unit price of interest rate risk, L, from the "down" probability and adding it to the "up" probability[8]. The l.h.s. of (9) is obviously the certainty-equivalent return of bond T.

Equation (8) has a more immediate interpretation in terms of the valuation of future cash flows. Let the current price of bond T be denoted as $B_0(T)$, which *increases* to $B_1^2(T)$ with probability p or decreases to $B_1^1(T)$ with probability $1 - p$ after one period. Replacing the bond returns $R_0^2(T)$ and $R_0^1(T)$ by $B_1^2(T) / B_0(T) - 1$ and $B_1^1(T) / B_0(T) - 1$ respectively, multiplying both sides of the resulting equation by the current bond price $B_0(T)$ and dividing both sides of the equation by $(1 + R_0)$ gives

$$
(9)\qquad B_0(T) = \frac{(p-L)\,B_1^2(T) + [1-(p-L)]\,B_1^1(T)}{(1+R_0)}\,.
$$

The bond price is equal to the end-of-period certainty equivalent cash flow of bond T discounted at the riskfree interest rate. This is of course a

[8] Note that "up" and "down" refers to the short interest rate and not to the bond T returns where it is opposite.

standard result. The equation however reveals how to include risk aversion into the binomial valuation model. “High” cash flows $B_1^2(T)$ (due to a low future interest rate) are “underweighted” and “low” cash flows $B_1^1(T)$ (due to a high future interest rate) are “overweighted” compared to a risk-neutral world where L, i.e. the premium to compensate for interest risk, is zero. This illustrates a very simple and intuitively appealing way to include risk aversion into the pricing of uncertain future cash flows caused by interest rate uncertainty.

The practical applicability of equation (9) is straightforward. $B_0(T)$ could be determined if $B_1^2(T)$ and $B_1^1(T)$ are known. However this is only true one period before maturity, where the bond price in all states can be determined by riskless discounting. Given these prices, formula (9) can then be applied to *recursively* determine all previous, and particularly, the current bond price. This corresponds exactly to the binomial stock option pricing model developed by *Cox / Ross / Rubinstein* (1979) except that the valuation is not preference-free. Numerical examples of the valuation model follow in Section 6.

5. Continuous Time Valuation

Merton (1974), *Brennan / Schwartz* (1977), *Vasicek* (1977), *Richard* (1978) and *Dothan* (1978) have developed arbitrage models for pricing the term structure of interest rates in a continuous time stochastic framework. Similarly *Cox / Ingersoll / Ross* (1985) have developed a general equilibrium term structure model in a continuous time consumption and production economy. It will be shown that the previous derivation can be easily related to the continuous time arbitrage equation.

First the expected discrete return on bond T is derived. It follows from equation (7) and from the definition of L that the expected return on bond T can be written as $E[R(T)] = R + L\,\sigma[R(T)]$, or expressed in our binomial terms

$$E[R(T)] = E[B_1(T)/B_0(T) - 1] = R_0 + L[R_0^2(T) - R_0^1(T)]. \tag{10a}$$

It was assumed that the future one period interest rate, R_1^j, $j = 1, 2$, is the only relevant source of uncertainty (i.e. factor risk) determining the stochastic properties of future interest rates; therefore we may write (10a) as

$$E[R(T)] = R_0 + L^* b_T \tag{10b}$$

where $L^* = L(R_1^1 - R_1^2)$ is the market price *per unit of short term* interest rate risk, and b_T is the riskiness of bond T with respect to the short term interest rate (see equation 6). Equation (10b) is an APT-like relationship because the expected return on any bond is the sum of the riskfree return plus a bond specific risk premium. In the limit of continuous interest rate changes the instantaneous bond return becomes

(11) $$E\,[R\,(T)] \;=\; E\,[dB\,(T)\,/\,B\,(T)] \;=\; (R \;+\; L^*\, b_T)\, dt$$

where $R\,dt$ is the instantaneous riskfree rate of return and L^* is the instantaneous market price per unit interest rate risk. Similarly b_t is the sensitivity of the bond return with respect to infinitesimally small interest rate changes and can be denoted by $b_T = d \ln B\,(T)\,/\,dR = [dB\,(T)\,/\,B\,(T)]\,/\,dR$. Substituting this expression into (11) and solving for the bond return in absolute (dollar) terms yields

(12) $$E\,[dB\,(T)] \;=\; [BR \;+\; L^*\, B_R\,(T)]\, dt$$

where $B_R\,(T)$ denotes the partial derivative of B with respect to R. Alternatively the dynamics of the bond price B can be expressed with Ito's Lemma. Applied to our one factor representation of the term structure, $B\,[R, t]$, this yields

(13) $$E\,[dB\,(T)] \;=\; B_R\,(T)\, E\,(dR) + \frac{1}{2}\, B_{RR}\,(T)\, \mathrm{Var}\,(dR) + B_t\,(T)\, dt\,.$$

Equating (12) and (13) leads to

(14) $$\frac{1}{2}\, B_{RR}\,(T)\, \mathrm{Var}\,(dR) + B_t\,(T)\, dt + B_R\, E\,(dR) - B\, Rdt \;=\; L^*\, B_R\,(T)\, dt$$

which is the well-known *arbitrage* bond valuation equation for a one factor representation of the term-structure derived by e.g. *Brennan / Schwartz* (1977) and *Cox / Ingersoll / Ross* (1985, 397[9]) and others. They demonstrate that a necessary (but not sufficient) condition supporting an arbitrage equilibrium is that L^* does not depend on the maturity of the bond (see equation 34 in their paper). In Section 2 we have imposed the even stronger restriction that L or L^*, respectively, is constant for all bonds. This condition may be weakened. Note however that maturity-independence is not sufficient for obtaining arbitragefree bond prices (see *Cox / Ingersoll / Ross* (1985, 398) for a counterexample). The case of a *constant* risk premium L^* however satisfies the no-arbitrage condition.

[9] Note that our reference is related to the *arbitrage* (not general equilibrium) equation of their paper. The equivalence is obvious after substituting $E\,(dR)$ by the drift term of the mean-reverting process $\alpha\,(r^* - r)\, dt$, Var (dr) by the variance of this process $\sigma^2 r\, dt$, and the risk premium L^* by $\lambda r\, dt$.

6. Examples

The application of the valuation formula in Section 3 is illustrated in this section. Instead of giving numerical formulae, which are immediate after the previous derivation, numerical examples are shown. They rely on the following interest rate progress:

$$
\begin{array}{llll}
R_0 = 0.05 & & & \\
R_1^1 = 0.06 & R_1^2 = 0.045 & & \\
R_2^1 = 0.07 & R_2^2 = 0.055 & R_2^3 = 0.04 & \\
R_3^1 = 0.08 & R_3^2 = 0.065 & R_3^3 = 0.05 & R_3^4 = 0.035
\end{array}
$$

The market price of interest rate risk, L, is arbitrarily set equal to 20%.

6.1 Arbitrage

Given these numerical values the price of a one period bond is given by 95.2381, and based on formula (9) we are able to compute the theoretical value of a 2 period discount bond as 90.2342 and the value of a 3 period discount bond as 85.0571. Suppose that the market price of the 2 period bond is 92.0000. Equation (3) tells how to efficiently arbitrage this mispriced bond. The number of $t = 2$ bonds, w^*, which is required to establish a riskfree position is

$$
(15) \qquad w^* = \frac{1}{1 + b_2/b_3} = \frac{1}{1 + \dfrac{R_0^2(2) - R_0^1(2)}{R_0^2(3) - R_0^1(3)}} = 2.1290
$$

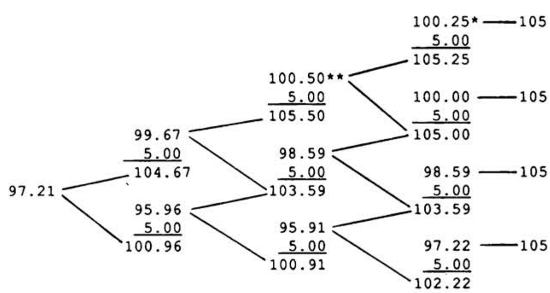
and the number of $T = 3$ bonds is given by $1 - w^* = -1.1290$. Thus a portfolio of 2.1290 units of bond $t = 2$ long and 1.1290 units of bond $T = 3$ short represents a perfect hedge against a one time interest rate change, or equivalenty perfectly mimics a riskfree investment. Alternatively bond $t = 2$ can be synthetically created by a long position of 1.1290/2.1290 = 0.5303 units of bond $T = 3$ and an investment of 1./2.1290 = 0.4697 units in one period riskless bond. This is illustrated in the following Table:

Arbitrage Position	Current Value	End of Period Value/ Interest Rate: 4.5%	6%
long 0.4738 B (1) @ 95.2381	− 45.12	+ 47.38	+ 47.38
short 1.0000 B (2) @ 92.0000	+ 92.00	− 95.69	− 94.34
long 0.5303 B (3) @ 85.0571	− 45.10	+ 48.31	+ 46.96
	+ 1.78	0.00	0.00

This illustrates the relative (or arbitrage) pricing of bonds. Given two future states of nature, two bonds with orthogonal payoffs (the 3-period and 1-period riskfree bond) completely span the state space of interest dependent payoffs and, hence, any other bond can be priced by arbitrage.

6.2 Coupon Bonds, Debt Options, Caps and Floors

Coupon bonds are, of course, easily valued by this procedure. Consider a callable 4 period, 5% coupon bond. The issuer has the option to call the bond after period 2 for 100.5 and after period 3 for 100.25. The following cash flow tree illustrates the pricing of the bond:



* Instead of 101.45, i.e. the bond is called.
* Instead of 101.03, i.e. the bond is called.

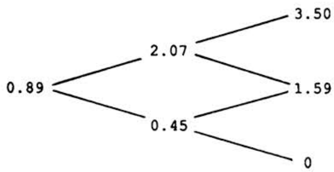
This also makes it possible to calculate bond volatilities (modified durations) for rather complex future cash flows. The traditional (duration based) "volatility" of this bond is calculated as

$$\text{Bond-Volatility} = -\frac{\text{Duration}}{1 + \text{Yield}} = \frac{3.71}{1.0575} = -3.508$$

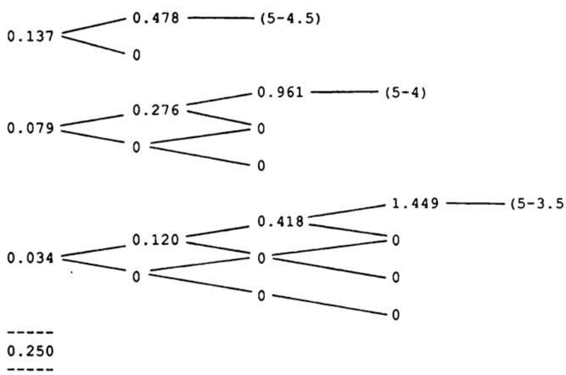
without taking into account the call provision. The binomial bond volatility is instead

$$\text{Binomial Bond Volatility} = -\frac{(99.67 - 95.96)/92.21}{0.06 - 0.045} = -2.68$$

which is substantially different. Obviously any kind of bond dependent claim can be priced by the same procedure[10]. For example an option to buy the previous bond for 102.00 after 2 periods is simply determined by



A special and growing type of debt options are interest rate caps and floors. Consider a 4 period 5%-floor. Since interest rates are set at the beginning of each period, the exercise decision is made at the beginning of each period, although the payoff from exercising the option occurs at the end of the respective period. For simplicity the several cash flows are valued separately:



(all figures in %)

i.e. the value of the floor is 250 basis points. The value of a cap can be determined in the same way.

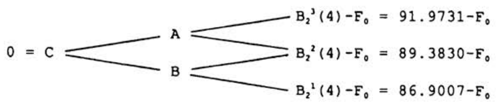
6.3 Futures Prices

Interest rate futures are priced equivalently. Recall that pricing a forward or futures contract is equivalent to determining a contractual part such that the present value of the contract is equal to zero (*Cox / Rubinstein* 1985, 60). The institutional difference between forward and futures contracts is widely

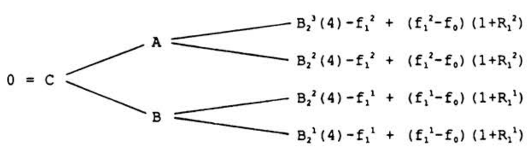
[10] A classical continuous time approach to the pricing of debt options is *Courtadon* (1980); an overview is provided by *Brennan / Schwartz* (1983).

known. For our purpose it is sufficient to recognize that, in the case of a forward contract, just one cash flow occurs at maturity of the contract (the difference between the prevailing bond price and the respective forward price) while in the case of a futures contract several cash flows occur during the life of the contract. Here we assume that the adjustment of the account (value change and interest) is executed after each (binomial) interest rate change.

Consider first a *forward* contract where the underlying instrument is a two period zero coupon bond. The contract expires in two periods. The forward rate is determined by the bond price ratio $\sqrt{B_0\,(4)\;/\;B_0\,(2)} - 1 = \sqrt{79.7735/90.2341} - 1 = 0.06354$. The forward price of a two period bond to be delivered in 2 periods is then $F_0 = 100/(1.06354)^2 = 88.4072$. Graphically, or translated to our binomial valuation model, the forward price is the value satisfying the following valuation tree:



This can easily be verified by inserting the respective value for F_0 (88.4072) and recursively evaluating C which must be equal to zero. This procedure is useful to visualize the derivation of the corresponding *futures* price f_0. The only difference between the forward price and the futures price arises from the fact that interest accrued on the account is added (or subtracted) periodically. Graphically this can be illustrated as follows:



with $B_2^3\,(4) = 91.7931$, $B_2^2\,(4) = 89.3830$, $B_2^1\,(4) = 86.9007$
$R_1^2 = 0.045$, $R_1^1 = 0.06$

Inserting these figures into the valuation tree, applying the recursive procedure and solving for f_0 yields, after some manipulations,

(16)

$$0 = \frac{(p - L)\,(1 + R_2^2)\,(f_1^2 - f_0)/(1 + R_2^2) + [1 - (p - L)]\,(1 + R_2^1)\,(f_1^1 - f_0)/(1 + R_2^1)}{(1 + R_0)}$$

$$= \frac{(0.3)\,(1.045)\,(f_1^2 - f_0)/1.045 + (0.7)\,(1.06)\,(f_1^1 - f_0)/1.06}{1.05}$$

which exhibits, beside the unknown futures price f_0, two other unknowns f_1^2 and f_1^1. Since in $t = 1$ there is just one period to expiration, there is no interest uncertainty and hence the futures price is equal to the forward price. f_1^1 can therefore be substituted by $F_1^1 = 87.6453$ and f_1^2 by $F_1^2 = 90.16$. Solving for f_0 gives 88.3996 which is slightly less than the forward price[11]. Generally the formula can be expressed as

$$f_0 = (p - L) f_1^2 + [1 - (p - L)] f_1^1 \tag{17}$$

which indicates that the same valuation procedure can be easily applied to futures pricing, except that discounting by the prevailing one period interest rate is omitted.

7. Discussion and Practical Issues

There are only a few binomial models for modelling the term structure of interest rates. A possible reason is discussed below. Similar to our approach, *Rendleman / Bartter* (1980) have modelled the evolution of the one period interest rates by a binomial process, but without modelling a term premium L. A different approach is taken by *Ho / Lee* (1987) who characterize the deviation of the actual future short rates from the forward rates as a binomial process. The arbitrage condition is then formulated in terms of arbitrage restrictions upon the evolution of the one period rates ("arbitrage-free rate movements"). By modelling the stochastic structure of the forward rates they are able to derive a preference-free valuation model, in the spirit of the *Black* (1976) model for pricing options on forward prices. In this sense our model is the binomial analogue to the continuous time approach by *Brennan / Schwartz* (1977), *Vasicek* (1978) and others and complements the work of *Ho / Lee* (1987).

While the main contribution of this paper is methodological, the question arises whether the model can be used for pricing interest contingent assets in practice. First we believe that this approach clarifies many problems specific to the pricing of assets whose payoffs depend on stochastic interest rates. It particularly clarifies the pricing of rather complex interest dependent payoffs as they could recently be observed on the bond market (droplock bonds, bunny bonds, FRN with caps and floors, fixed-floating conversion options, swaptions, and others).

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[11] See *Cox / Ingersoll / Ross* (1981, 332), for a formal proof for this condition. Basically the futures price is less than the forward price if the covariance between the underlying bond and a discount bond (representing the interest rate risk) is less than the variance of the discount bond, $\sigma[P, V] > \sigma^2[V]$. Given our term structure assumption, the covariance is equal to $\sigma[P, V] = \sigma[P]\,\sigma[V]$. In our case the underlying bond has a duration of 2 years and the "discount bond" is a one period investment, i.e. $\sigma[P] > \sigma[V]$. Therefore, the condition is satisfied.

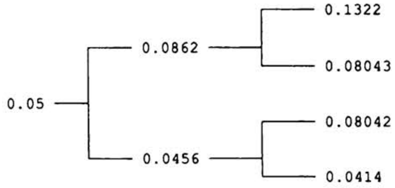
Second, some empirical issues may be relevant for practical applications. It has become standard in the theoretical literature to model interest rates as stationary, mean-reverting processes. Most prominently a mean-reverting square-root process is proposed. According to this model, the interest rate dynamics is characterized by

$$dr = \alpha (r^* - r)\, dt + \sigma \sqrt{r}\ \widetilde{dz} \tag{18}$$

which is a first-order autoregressive process in continuous time. $\widetilde{dz}$ is a Standard Wiener process. The interest rate is elastically pushed towards a long term interest rate r^* if $a > 0$ and $r^* > 0$; α determines the speed of the convergence of the process. The properties of this process are discussed in *Cox / Ingersoll / Ross* (1985, 391). The discrete analogue of this process can be written as[12]

$$r(t+1) = [k \sqrt{r(T)} + d + \tilde{u}]^2 \tag{19}$$

where $r(T)$ is the simple interest rate in t for period $[t; t + 1]$, k is the autocorrelation coefficient and d the mean of the series. $\tilde{u}$ is a stationary random process with mean zero and variance v^2. If u takes $u = \pm v$ and the parameters are specified as $v = 0.04$, $d = 0.03$, $k = 0.9$ and $r(0) = 0.05$ then the following binomial process results:



which is just to indicate that the evolution of interest rates becomes rather complex given a commonly used interest rate representation[13].

There is however some doubt whether this type of process is really empirically justified. It can even be questioned whether interest rate processes are stationary (mean-reverting). At least for the most recent time period, Dickey-Fuller unit root tests cannot reject the zero hypothesis of nonstationary interest rates on the 1 % and 5 % confidence limit[14]. Therefore at least the complications arising from the mean-reverting property of the process could

[12] See *Fischer / Zechner* (1984) for examples of discrete representations of continuous time interest rate processes and the literature cited therein.

[13] The tree of short interest rates is separated after each interest rate change. With monthly interest rates and only 2 years to maturity, the number of interest rate changes is 23 and there are about 8 millions (2^{23}) interest rates at the end of the tree.

[14] See *Planta* (1989) or *Zimmermann* (1988) for some respective evidence.

be avoided by using a "closed tree" interest rate process[15]. The second difficulty is to quantify the ex ante risk premium L and the probabilities of the interest rate movements (i.e. the expected rate of change of interest rates). These are however problems specific to *all* valuation approaches modelling the stochastic process of *short* rates and need therefore not to be discussed here. In summary we believe that the binomial approach provides the same advantages in pricing interest contingent assets as the *Cox / Ross / Rubinstein* (1979) binomial model in pricing stock options.

Summary

The binomial option pricing approach of Cox / Ross / Rubinstein (1979) is applied to the pricing and hedging of interest rate contingent assets. An arbitrage based valuation formula is derived and applied to the pricing of coupon bonds, bond options and futures.

Zusammenfassung

Das binomiale Preisbildungsmodell von Cox / Ross / Rubinstein (1979) wird verwendet, um zinsabhängige Finanzanlagen zu bewerten. Die resultierende Preisbildungsformel beruht auf Arbitrageüberlegungen und wird illustrativ zur Preisbildung von Kupon-Obligationen, Bond-Optionen und -Futures herangezogen.

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[15] Note that even if the process would be "slightly" mean-reverting, the error from using a "closed tree" would be probably small; see for instance the previous numerical example, where two final nodes exhibit almost the same numerical values.

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