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Decision Costs and Microeconomic Demand For Money

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In the context of the microeconomic demand for money this paper analyzes a logical problem that arises when it is assumed that all decisions are costly. This difficulty has surfaced in the hypothesis that „super-optimization“, i.e. optimization taking all costs into consideration, is impossible when calculations are costly.

The problem is set in a continuous time environment, and it is assumed that individual goods prices are stochastic. The paper contains a discussion of (a) the gains from making new decisions at any time *t*, and (b) the impacts of price inflation or deflation on the household's demand for money.

1. Introduction

This essay is a continuation of a previous paper „Computational Costs and Bounded Rationality“ (*Gottinger*) (1981)) as an application of the assumption of costly decision making to the behavior of the household in determining its pattern of consumption. The problem is set in a continuous time environment, and it is assumed that individual goods prices are stochastic. Since decisions are costly to produce, a household will not make a new decision about the basket of goods to purchase and about the form of its wealth holdings if the environment continues to yield data which are close to the expectations from which the old decision was produced. The production of a decision is therefore an investment, and an agent will not plan to replace an old decision until the expected value of a new decision exceeds its expected cost. It is assumed that the household determines the appropriate time to make a new decision according to the level of its money balances. Since decisions are not continuously revised and since the environment is expected to change during the period in which a given decision is in effect, a demand for money of an inventory type arises. The effects on this money demand of changes in the rate of inflation are examined. In particular, the model can be specialized to an inflationary period in which incomes are indexed and in which the inflation rate is stochastic[1].

Consider a choice problem facing a consumer for whom decision making is costly in that the computation of a decision vector of goods consumption

[1] The model appears to be close in spirit to Friedman's work on the demand of money [*Friedman* (1953), (1959)].

and of asset holdings based on price information requires the use of some scarce resource such as time. I assume that the consumer holds money as a signaling device to determine when it is time to make a new decision. The consumer monitors his cash holdings until they reach some bound and then makes a new decision on his behavior for the future.

This approach differs from that of *Gottinger* (1981) in which each decision required the choice of a separate decision algorithm. Here it is assumed that the decision technique remains unchanged for each new decision; and the means by which the consumer determines the particular decision technique is not considered. Also, the problem considered here is not a single decision; rather, it is a dynamic problem in which the household makes decisions at discrete intervals over an infinite horizon.

The decision process used in this paper is similar to that followed by the *Miller-Orr* (1966) firm which holds cash to substitute away from transactions costs. However, the consumer in this case uses money to substitute away from the cost of decision making.

In Section 2. I list the assumptions of the model and set up the problem in a continuous time framework. In Section 3. I derive implications on the form of the household's demand for money. Section 4. relates the model to a simple type of partial adjustment models. The appendix derives technical results used elsewhere in the text.

2. Assumptions of the Model

The model is based on the following assumptions:

a) The consumer's utility function $U: R^n \to R$ is concave, and the consumer has a subjective rate of discount ϱ.

b) Goods prices P_i, $i = 1, 2, \ldots, n$, follow the continuous stochastic process

$$dp_i / p_i = \eta_i \, dt + \sigma_i \, dz_i, \; i = 1, 2, \ldots, n \tag{1}$$

where $\{ z_i(t) \}_{t=0}^{\infty}$ is a Wiener process.

c) A money income of $y(t)$ is earned and received instantaneously from the sales of a fixed amount of labor and from the return on earnings assets; $y(t)$ is assumed to be continuous and it may be random.

d) Initial nominal non-money wealth $W(t)$ consists of an asset which earns an instantaneous nominal rate of return $r(t)$, which may be random.

e) There are no transactions costs incurred in purchasing goods or in converting assets.

f) A lump sum nominal cost $c(t)$ is incurred in simultaneously deciding the optimal vector of goods to consume and the optimal portfolio of assets.

g) All price information is freely available.

The consumer's problem is to maximize the discounted stream of expected utility using consumption vectors and money holdings as controls. Since decision making entails a lump sum cost the consumer will not revise his decisions continuously. Rather, he will decide on goods and asset holdings at finite time intervals, the length of which will depend on the particular decision rule that the consumer chooses to signal that it is time to make a new decision. I also assume that

h) The consumer makes a new decision on bundies of goods to consume and on money holdings whenever $M(t) \leq 0$ or $M(t) \geq \varphi(\underline{p}(t))$ where $\varphi(\cdot)$ is some function of the prices p_i, $i = 1, 2, \ldots, n$; and $M(t)$ is the amount of nominal balances held at time t; $\varphi(\cdot)$ is given to the consumer (or chosen in some earlier decision).

Assumption (h) implies that a change in the money asset requires no costly decision – only a decision to make a change in earning assets and in the goods vector is costly. Money is useful to the household because it provides a means of avoiding the costs of decision making.

The use of money by the household as a signaling mechanism seems fairly natural. When money balances run out, some action must be taken to change assets or consumption baskets. Also, there must be some method available to the household which allows it to face unexpected changes in receipts and expenditures without making a revision in its previous decision. The holding of money in this model plays this role, and the reason for holding money appears similar to the usual precautionary motive except that the cost of illiquidity takes the form of the cost of decision-making. However, the household can never be illiquid in this model because a failure to pay for consumption requires a change in the earning assets of the household, and this in turn requires a new decision. Therefore, money holding is not primarily a means of insuring against illiquidity because the household can never spend more money than it has. It is primarily an indicator for new decisions, and the implicit zero cost in decision resources for its use makes it a useful signaling device.

The upper bound on money holdings is present to allow the household to make a new decision if its cash inflow is consistently greater than its outflow. Thus $\varphi(\cdot)$ may be considered as a kind of price index which is used to evaluate the information of upward movements in money holdings in terms of their function as a signaling device. In the event of a proportional change in prices and wealth, nothing real should change, including the distribution of the passage time of money through its upper and lower bounds. Therefore, $\varphi(\cdot)$ should be homogenous of degree 1 in prices and wealth.

Formally, the household's problem is to calculate the polity functions for the functional equation

$$\begin{aligned} V[w(t), \underline{p}(t), y(t), \underline{x}, M(t)] \equiv\ & E_t \{\ U(\underline{x}) \int_t^{t_1} e^{-\varrho t} = dt + \\ & + e^{-\varrho(t_1 - t)} \max_{\underline{x}^*, M^*} V[w(t_1) - C(t_1) + M(t_1) - M^*; \underline{p}(t_1), y(t_1), \underline{x}^*, M^*]\} \end{aligned} \tag{2}$$

where $y(t)$ is nominal income at time t, $\underline{p}(t)$ is the price vector of goods at time t, $\underline{x}$ is the bundle of goods consumed at time t, and $V(\cdot)$ is an unknown value function.

The expectation is based on the c.d.f. for t_1, the time at which nominal money holdings pass through their upper or lower bounds. The c.d.f. for t_1 is

$$\psi(t) \equiv [1 - \phi(t)] \equiv 1 - Pr\{0 \leq M(\tau) \leq \varphi(\underline{p}(\tau)), \forall\ \tau \leq t\}\,. \tag{3}$$

The sequence of events is as follows. At time t, the household consumes a basket $\underline{x}$ and has money holdings $M(t)$, determined in the latest decision and in the random movement of net expenditure in the interim. As time passes, the household monitors its money holdings while it continuously buys the bundle of goods $\underline{x}$ regardless of the movement of prices. When the household's money balances first pass out of the bounds set on nominal balances at time t_1, a new decision is made. Accordingly, a lump sum cost $c(t_1)$ is paid out of wealth; and non-money assets are increased by the difference between the household's money holdings at the time of the new decision, $M(t_1)$, and those determined by the new decision M^*.

The continuous time problem of the household can be viewed as a discrete time problem whose time intervals are random variables determined by household choice. The function $\psi(t)$ which is dependent on $\underline{x}$ and $M(t)$, is a first passage c.d.f.; and, therefore, an explicit formulation of ψ for even the simple process on $\underline{p}$ assumed here is difficult to produce. The p.d.f.'s for first passage problems are usually determined explicitly for a very narrow class of random variables, those based on the simple Wiener process. The generation of explicit p.d.f.'s for first passage times of more complex processes does not seem to produce enough of a reward to offset the computation cost of deriving proofs since they are scarce in the literature. For derivations of some first passage p.d.f.'s, see *Feller* (1971). *Miller* and *Orr* (1966) exploited a first passage p.d.f. derived in Feller in developing their inventory model for money holding.

The properties of the policy functions for (2) seem to be out of reach because of the difficulty in finding a first passage p.d.f. However, in this paper, I am particularly interested in finding the effects of the parameters on $M(t)$. Since $M(t)$ is almost always assumed to react passively to price and income changes, it seems possible to determine a functional relationship between $M(t)$ and the underlying parameters of the model. Specifically,

$$M(t) = M(t_0) + \int_{t_0}^{t} y(t)\,dt - \int_{t_0}^{t} \underline{p}(\tau)'\,\underline{x}\,d\tau \tag{4}$$

given that a decision is made at $t_0 < t$ and no new decision is made through time t. The expected value of $[M(t) - M(t_0)]$ is the expected value of the two stochastic integrals on the right hand side of (4) conditional on there having been no new decision in the time period. However, the money stock may pass through its barriers any number of times in a finite time interval because more than one decision may be made; so without an explicit formula for the first passage c.d.f., $E_{t_0}\,\Delta M(t)$ is difficult to calculate. Therefore, this problem will be reformulated with the aim of deriving a relationship between changes in the money holdings of the household and the underlying parameters during infinitesimal increments of time.

In order to do this (2) will be converted into a more useful form. It can be shown (see Appendix a) that

$$\begin{aligned} & V[w(t), y(t), \underline{p}(t), \underline{x}\,M(t)] = \phi(k)\left\{ U(\underline{x}) \left[\frac{1}{\varrho}(1 - e^{-\varrho k}) \right] + \right. \\ & \left. + E_t^*\, e^{-\varrho k}\, V[w, y(t+k), \underline{p}(t+k), \underline{x}, M(t+k)] \right\} + [1 - \phi(k)]\,[U(\underline{x}) \\ & \frac{1}{\varrho}(1 - e^{-\varrho\lambda k}) + e^{-\varrho\lambda k} E_t^{**} \max_{x^*, M^*} V[W - c + M(\lambda k) - \\ & - M^*, \underline{y}(t+\lambda k), \underline{p}(t+\lambda k), x^*, M^*] + o(k) \equiv \phi(k)\, Z_1(k) + \\ & + [1 - \phi(k)]\, Z_2(\lambda k) + o(k)\,. \end{aligned} \tag{5}$$

The value function V equals the expected value of the two contingencies: either money holdings remain inside their bounds with probability $\phi(k)$ during interval k or they pass out of their bounds with probability $[1 - \phi(k)]$ during the interval k. In the former case no new decision is made, and the goods consumption basket and earning assets W remain unchanged. In the latter case, a new decision is made on the goods basket and on nominal money holding in order to maximize V. The possibility of passing through the barriers more than once, given that a decision is made in the interval, is ignored because the probability is $o\,(k)$. The expectations E_t^* and E_t^{**} are conditional on not having passed through the barriers and on having passed through the barriers, respectively. If the first passage takes place in the interval, it is assumed that it occurs at time $t + \lambda k$ where $0 < \lambda < 1$.

Expanding the RHS of (5) around $k = 0$,

$$(6) \qquad V(\cdot) = \phi(0)\, Z_1(0) + \phi'(0)\, Z_1(0)\, k + \phi(0)\, Z'(0)\, k + + [1 - \phi(0)]\, Z_2(0) - \phi'(0)\, Z_2(0)\, k + [1 - \phi(0)]\, Z_2'(0)\, \lambda k + o\,(k)\,.$$

Since $\underline{p}\,(t)$, $\underline{x}'\,\underline{p}\,(t)$, $y\,(t)$, and $M\,(t)$ are continuous with probability one, $\phi(0) = 1$ because if $M\,(t)$ is within its boundaries it can make a discontinuous jump out of its boundaries only with zero probability. Therefore, substituting for z_1 and z_2 from (5) (see Appendix b for more detailed calculations),

$$(7) \qquad \begin{aligned} 0 = {} & V(t) + \phi'(0)\, V(t)\, k + \{\, U(\underline{x})\, k - \varrho V(t)\, k + V y E_t^* \Delta y + \sum_{i=1}^{n} V_{p_i} E_t^* \Delta p_i + \\ & + V_M E_t^* \Delta M + \frac{1}{2} [E_t^* \Delta \underline{p}' V_{\underline{p}\underline{p}} \Delta \underline{p} + E_t^* \Delta \underline{p}\, V_{\underline{p}y} \Delta y + E_t^* V_{yy} (\Delta y)^2]\} - V(t) - \\ & - \phi'(0)\, E^{**} \max_{\underline{x}^*, M^*} V[W - c\,(t) + M\,(t) - M^*, y\,(\mathrm{t}), \underline{p}\,(t), \underline{x}^*, M^*] + o\,(k) \end{aligned}$$

where all partials are evaluated at time t. Since $M\,(t)$ is a process that may jump discontinuously, one must wonder if the value function will be differentiable. It is clear that $M\,(t)$ behaves like a combination of a continuous diffusion process and of a jump process. $M\,(t)$ moves continuously for almost all t and only jumps at discrete intervals; and the probability of a jump depends on the probability of first passage of the money process. Therefore, there should be no problem with the differentiability of V. See *Dreyfus* (1965) for a discussion of the differentiability of V when it is a function of a jump process.

Dividing (7) by k and taking the limit as $k \to 0$,

$$(8) \qquad \begin{aligned} 0 = {} & \phi'(0)\, V(t)\, U(\underline{x}) - \varrho V(t) + V_y \eta_y + \sum_i V_{p_i} \eta_i p_i + V_M [y\,(t) - \underline{x}'\, \underline{p}\,(t)] + \\ & + \frac{1}{2} S\,[V(t)] - \phi' \max_{x^*, M^*} V[W(t) - c\,(t) + M\,(t) - M^*, y\,(\mathrm{t}), \underline{p}\,(t), \underline{x}^*, M^*] \end{aligned}$$

where $S\,(V)$ is the sum of the second order terms in (7). Equation (7) follows from (6) because $[1 - \phi(0)] = 0$; $\dfrac{1}{\varrho}(1 - e^{-\varrho k}) = 0$ at $k = 0$; and exp

$(-\varrho k) = 1$ at $k = 0$. The second partial terms must be carried along because the variance of an element of a Wiener process is of order k. The means $E_t^* \Delta y$, $E_t^{**} \Delta p_i$, $i = 1, \ldots, n$, and $E_t^* \Delta M(t)$, are also of order k.

From (8), we have

$$\begin{aligned}
&\max_{\underline{x}^*, M^*} V[W - c(t) + M(t) - M^*, y(t), \underline{p}(t), \underline{x}^*, M^*] - \\
&- V[W(t), y(t), \underline{p}(t), \underline{x}, M(t)] = \\
&= \frac{1}{\phi'(0)} \{ U(\underline{x}) - \varrho V(t) + V_y \eta_y + \sum V_{p_i} \eta_i p_i + \\
&+ V_M [y(t) - \underline{x}' \underline{p}(t)] + \frac{1}{2} S[V(t)] \}.
\end{aligned} \tag{9}$$

Equation (9) expresses in terms of utility the cost of making a new decision at time t. If the LHS of (9) is less than zero, then the expected net benefit of a new decision is outweighed by its cost and the decision should not be made.

Equation (9) holds for any time t. In particular, it holds the instant after a new decision is made at some time t_0. If

$$\begin{aligned}
G_{t_0}(t) \equiv &\max_{\underline{x}^*, M^*} V[W(t_0) - c(t) + M(t) - M^*, y(t), \underline{p}(t), \underline{x}^*, M^*] - \\
&- V[W(t_0), y(t), \underline{p}(t), \underline{x}, M(t)]
\end{aligned} \tag{10}$$

then $G_{t_0}(t)$ is the gain of making a new decision at time $t > t_0$, given that no decisions have been made between t_0 and $t. G_{t_0}(t)$ is a random variable whose distribution is conditional on information available at time t_0.

If the decision based on the movement of nominal balances is optimal, it should function so that the distribution of the first passage time of money balances starting at time t_0 is the same as the distribution for the first passage time at which the process $G_{t_0}(t) = 0$.

More formally, assume that $\psi'_{t_0}(t) \equiv -\phi'_{t_0}(t)$ exists where $\psi'_{t_0}(t)$ is the first passage p.d.f. for money holdings based on information available immediately after a new decision $(\underline{x}^*, M^*)$ at time t_0. Also, assume that $H'_{t_0}(t)$ exists where $H'_{t_0}(t)$ is the first passage p.d.f. of $G_{t_0}(t)$ through zero based on information available at time t_0. Then $\psi'_{t_0}(t)$ should equal $H'_{t_0}(t)$, $\forall\, t$.

This can be shown by considering equation (9). When $G_{t_0}(t) = 0$, the first passage of $G_{t_0}(t)$ has occurred and it is beneficial for the household to make a decision. For $G_{t_0}(t)$ to equal zero, it is clear that either $\phi'(0) = -\infty$ or $[U(\underline{x}) - \varrho V(t) + V_y \eta_y + \Sigma V \varrho_i \eta_i \varrho_i + \ldots] = 0$. If it is expected at time t_0 that $\phi'(0) = -\infty$ when $G_{t_0}(t) = 0$, then $H'_{t_0}(t) = \psi'_{t_0}(t)$, $\forall\ t$, because the events of first passage of $M(t)$ and of $G_{t_0}(t)$ are expected to occur simultaneously at time t_0. Therefore, their first passage p. d. f.'s will be identical.

Suppose, on the other hand, that at time t_0 it is expected that $\phi'(0) \neq -\infty$ when $G_{t_0}(t) = 0$. The first passage of $G_{t_0}(t)$ and of money balances are not expected to coincide, so $\psi'_{t_0}(t) \neq H'_{t_0}(t)$ for some t. Let t_1 be such that $\psi'_{t_0}(t_1) > H'_{t_0}(t_1)$. Then it is considered more likely that a decision will be made at t_1 than a decision at time t_1 will be beneficial. But then $(\underline{x}^*, M^*)$ is not an optimal choice at time t_0 since this choice makes it likely that a decision will be made at an unfavorable time. Hence, the household expects that $\phi'(0) = -\infty$ when $G_{t_0}(t) = 0$ based in information available at time t_0. This is another way of saying that money holdings and expenditure rates are arranged so that the money indicator is *ex ante* a good device for signaling a new decision time.

To consider informally the effect of a general fall in the rate of increase of prices on M^*, the optimal choice of nominal balances, I specialize the model to the one good case, I also assume that income is indexed so that $y(t) = \bar{y}p(t)$ for some constant $\bar{y}$.

Suppose that a decision to set M^* and x^* is made at time t_0 based on the expected rate of inflation η_1; $\psi'_{t_0}(t)$ should be identical to the first passage p. d. f. of $G_{t_0}(t)$ through zero.

Now suppose instead that at time t_0, η_2 is expected where $\eta_2 < \eta_1$ and that M_1^* and x_1^* are fixed at the levels determined for η_1. If the agent had considered it more likely that his money would first pass through the lower bounds based on expectation η_1, he will now expect that his money holdings will have a later time of first passage because of the reduction in the expenditure rate, i. e.,

$$\int_{t_0}^{\infty} \tau\, \psi'_{t_0}(\tau \mid \eta_1, x_1^*, M_1^*) < \int_{t_0}^{\infty} \tau\, \psi'_{t_0}(\tau \mid \eta_2, x_1^*, M_1^*)\,.$$

The lower η_2 causes $W[W(t), y(t), P(t), M_1^*(t), x_1^*]$ to be larger, $\forall\ t$, because of the decreased cost of nominal money holding. Since the choice of (M_1^*, x_1^*) is not optimal for η_2, $\max\limits_{\hat{x}, \hat{M}} V[W(t) - c(t) + M(t) - M, y(t), P(t), \hat{M}, \hat{x}]$ should rise by relatively more than $V[W(t), y(t), P(t), M_1^*(t), x_1^*]$, $\forall\ t$, because an optimal decision at any time t is relatively more valuable than before. Then the first passage p. d. f. for $G_{t_0}(t)$ should have a smaller expected value. However, an optimal choice for the new η_2 would cause both first passage p. d. f.'s to coincide; so the agent must act to increase the expected value of the first passage time for $G_{t_0}(t)$ and to reduce that of his money holdings.

Since the substitute for decision making is now relatively cheaper, there should be a reduction in the number of decisions made per period or an increase in the mean time between decisions. Therefore, when M^* and x^* are adjusted for the new η_2, $\int_{t_0}^{\infty} \tau\, \psi'_{t_0}(\tau \mid \eta_2)$ will not be reduced below the old mean decision time.

The adjustment in the first passage p.d.f. is effected through changes in x^* and M^*. Since the household is better off, x^* should rise; and since the cost of real balances is reduced relative to the cost of decision making more real balances should be held on the average during the decision period. The rise in x^* increases the rate of expenditure and reduces the mean between decisions so that

$$\int_{t_0}^{\infty} \tau\, \psi'_{t_0}(\tau \mid \eta_2) < \int_{t_0}^{\infty} \tau\, \psi'_{t_0}(\tau \mid \eta_2, x_1^*, M_1^*)\,.$$

Whether the optimal M^* for η_2 rises or falls with the reduction in the expected inflation rate depends upon $\varepsilon_{x^*,\,\eta}$, the elasticity of x^* with respect to η. It also depends upon M_1^*, η_1, and $P_{t_0} x_1^*$, the optimal money holdings based on η_1, the expected rate of inflation η_1, and the initial optimal rate of expenditure associated with η_1, respectively.

This can be supported by assuming that $\varepsilon_{x^*,\,\eta} < 0$, that $\eta < 0$, and that $x^* - \bar{y} > 0$. This latter assumption is consistent with the assumption that money is expected to pass through the lower bound first. The following argument will be based on the assumption that *prices* follow their expected path rather than money holdings.

Suppose that M^* and x^* are optimal decisions for the expected instantaneous inflation rate η. If prices follow their expected path then

$$M(t) = M^* + \int_{t_0}^{t} EP(\tau)\,(\bar{y} - x^*)\, d\tau = M^* + P_0\,(\bar{y} - x^*)\,\{\exp[\eta\,(t - t_0)] - 1\}\,/\,\eta$$

or

$$\text{(11)} \qquad \frac{M^*\,\eta}{P_0\,(x^* - \bar{y})} + 1 = \exp[\eta\,(\hat{t} - t_0)]$$

where $\hat{t}$ is the time that the path of money passes through zero. Taking logarithms of both sides of (11)

$$\text{(12)} \qquad \frac{1}{\eta} \log \left[\frac{M^*\,\eta}{P_0\,(x - \bar{y})} + 1 \right] + t_0 = \hat{t}\,.$$

Taking the derivative of both sides of (12) with respect to η,

(13)
$$-\frac{1}{\eta^2}\log(Z+1)+\frac{1}{\eta}\frac{1}{(Z+1)}\left|\frac{\dfrac{\partial M^*}{\partial\eta}\eta}{P_0(x^*-\bar{y})}+\frac{M^*}{P_0(x^*-\bar{y})}-\frac{Z}{(x-\bar{y})}\frac{\partial x^*}{\partial\eta}\right|=\frac{\partial\hat{t}}{\partial\eta}<0$$

where $z = M^*\eta / P_0 (x^* - \bar{y})$. The inequality follows from the above discussion about the movement of the expected time of first passage. Multiplying throug by η^2 and shifting and cancelling terms

(14)
$$\frac{Z}{Z+1}\left[\varepsilon_{M^*,\eta}+1-\frac{x}{(x-\bar{y})}\varepsilon_{x^*\eta}\right]<\log(Z+1)$$

where $\varepsilon_{M^*,}\eta$ and $\varepsilon_{x^*,}\eta$ are the elasticities of optimal money holdings with respect to η and of optimal goods consumption with respect to η, respectively. Then,

(15)
$$\left[\varepsilon_{M^*,\eta}+1-\frac{x^*}{(x^*-\bar{y})}\varepsilon_{x^*,\eta}\right]<\frac{Z+1}{Z}\log(Z+1)\,.$$

For small enough z, $\log(z+1) = z$ and the inequality in (15) still holds so

(16)
$$\varepsilon_{M^*,\eta}-\frac{x^*}{(x^*-\bar{y})}\varepsilon_{x^*\eta}<\frac{M^*\eta}{P_0(x-\bar{y})}$$

or

(17)
$$\varepsilon_{M^*\eta}<\frac{M^*\eta+P_0x^*\varepsilon_{x^*\eta}}{P_0(x-\bar{y})}\,.$$

Then $\varepsilon_{M^*\eta} < 0$ if $\varepsilon_{x^*\eta} < -M^*\eta / P_0 x^*$. The sign of $\varepsilon_{M^*\eta}$ depends on whether $\varepsilon_{x^*\eta}$ is less than or greater than $-M^*\eta / P_0 x^*$. Since a change in the inflation rate produces a wealth effect only through the change in value of the money asset of the household, one would expect the magnitude of $\varepsilon_{x^*\eta}$ to be small. Presumably, $M^* > P_0 x^*$ since the money that the household chooses to hold should be greater than the instantaneous flow of expenditure. For low expected instantaneous rates of inflation η is small so $\varepsilon_{x^*\eta} < -M^*\eta / P_0 x^*$ and $\varepsilon_{M^*,\eta} < 0$. For high η, $\varepsilon_{M^*\eta} > 0$.

It should be noted that if $\varepsilon_{M^*\eta} > 0$, there is no implication that the expected average real balances held during the decision period will fall with a fall in η even though expected average nominal balances will fall.

3. A Specification of Money Demand

In this section I specialize the analysis to determine a specification for the demand for money by the household. This falls fairly neatly out of the assumption on the use of money in the household's decision rule and out of the description of the previous section.

The household is assumed to make a decision on new nominal money holdings and on goods consumption when $M(t) \leq 0$ or $M(t) \geq \varphi(\underline{p})$ where φ is given.

From (4), the change in "real" balances during the period of length k is

(18)

$$\Delta \frac{M(t)}{\varphi(\underline{p})} \equiv \frac{M(t+k)}{\varphi[\underline{p}(t+k)]} - \frac{M(t)}{\varphi[\underline{p}(t)]} = \phi(k)\left(\frac{M(t) + \int_{t_0}^{t+k} y(\tau)\,d\tau - \int_{t}^{t+k} \underline{p}'(\tau)\,\underline{x}\,d\tau}{\varphi[\underline{p}(t+k)]}\right)$$

$$+ [1-\phi(k)]\left(\frac{M^* + \int_{t_0}^{t+k} y(\tau)\,d\tau - \int_{t_0}^{t+k} \underline{p}(\tau)'\,\underline{x}^*\,d\tau}{\varphi[\underline{p}(t+k)]}\right) - \frac{M(t)}{\varphi[\underline{p}(t)]} + o(k)$$

where $o(k)$ encompasses the effects of two or more decisions during k. Real balances are measured by using $\varphi(\underline{p})$ as a deflator. Since the class of functions from which φ is selected has not been specified, the usual price indexes used as deflators have not been excluded by the scheme. However, more general deflators are allowed. Since the deflator is chosen by the consumer to set a first passage p.d.f. and not to measure the change in cost of maintaining a certain level of well-being, it is likely that the deflator will be different from the CPI.

A more general scheme would allow the consumer to select $\varphi(\cdot)$ at the same time as x^* and M^*. For example, $\varphi(\ )$ could be restricted to the set of weighted averages of prices, i.e. $\varphi(\underline{p}) = \sum_{i=1}^{n} w_i p_i$ for some w, $w_i \geq 0$, $\forall\ n$ and $\sum_{i=1}^{n} w_i = 1$. Then the household will use $(\underline{w}^*, \underline{x}^*, M^*)$ as a control, and the deflator by which it measures the real value of its nominal balances is selected to maximize $V(\cdot)$.

Equation (18) can be expanded in a Taylor series to yield (see Appendix c),

$$(19)\qquad \Delta \frac{M(t)}{\varphi[\underline{p}(t)]} = \frac{y(t) - \underline{x}'\,\underline{p}(t)}{\varphi[\underline{p}(t)]}\,k - \frac{M(t)}{\varphi[\underline{p}(t)]}\,\frac{\Delta\varphi(p)}{\varphi(\underline{p})} +$$

$$+\ \phi'(0)\left[\frac{M(t)}{\varphi(\underline{p})} - \frac{M^*}{\varphi(\underline{p})}\right]k + \frac{M(t)}{\varphi(\underline{p})}\left[\frac{\Delta\varphi(\underline{p})}{\varphi(\underline{p})}\right]^2 + o(k)$$

if $y(t)$ is assumed to be non-stochastic. The term in $(\Delta\varphi/\varphi)^2$ must be carried along because this term is of order k.

By Ito's lemma,

$$\Delta \varphi [\underline{p}(t)] = \sum_{i=1}^{n} \varphi_i \, \Delta p_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi_{ij} \, \Delta \underline{p}_i \, \Delta \underline{p}_j' + o(k) \tag{20}$$

where φ_i and φ_{ij} are first and second partial derivatives of φ, respectively. But $\Delta p_i \, \Delta p_j = \sigma_i \, \sigma_j \, p_i \, p_j \, \varrho_{ij} \, k$ by assumption (b) in Section 2, where ϱ_{ij} is the correlation between z_i and z_t. Then

$$\Delta \varphi [\underline{p}(t)] = \left[\Sigma \, \varphi_i \, \eta_i \, p_i + \frac{1}{2} \Sigma \, \Sigma \, \varphi_{ij} \, \sigma_i \, \sigma_j \, p_i \, p_j \, \varrho_{ij} \right] k + \Sigma \, \varphi_i \, \sigma_i \, dz_i + o(k)\,. \tag{21}$$

Substituting (21) into (19) yields

$$\begin{aligned} \Delta \frac{M(t)}{\varphi[\underline{p}(t)]} &= \frac{y(t) - \underline{x}' \, \underline{p}(t)}{\varphi(\underline{p})} k - \\ &- \frac{M(t)}{\varphi(\underline{p})} \frac{\left[\Sigma \, \varphi_i \, \eta_i \, p_i + \frac{1}{2} \Sigma \, \Sigma \, \varphi_{ij} \, \sigma_i \, \sigma_j \, p_i \, p_j \, \varrho_{ij} \right] k + \Sigma \, \varphi_i \, \sigma_i \, p_i \, dz_i}{\varphi(\underline{p})} + \\ &+ \phi'(0) \left[\frac{M(t) - M^*(t)}{\varphi(\underline{p})} \right] k + \frac{M(t)}{\varphi(\underline{p})} \frac{\Sigma \, \Sigma \, \varphi_i \varphi_j \, p_i \, p_j \, \sigma_i \, \sigma_j \, \varrho_{ij}}{\varphi(\underline{p})^2} k + o(k)\,. \end{aligned} \tag{22}$$

Dividing by $M(t) \,/\, \varphi(\underline{p})$, taking expected values, and taking the limit as $k \to 0$,

$$\begin{aligned} E\left(\frac{d\,M/\varphi}{M/\varphi} \right) = \Bigg\{ & \frac{y(t) - \underline{x}' \, p(t)}{M(t)} - \frac{\Sigma \, \varphi_i \, \eta_i \, \varrho_i}{\varphi(\underline{p})} - \frac{1}{2} \frac{\Sigma \, \Sigma \, \varphi_{ij} \, p_i \, p_j \, \sigma_i \, \sigma_j \, \varrho_{ij}}{\varphi(\underline{p})} + \\ & + \phi'(0) \left[1 - \frac{M^*}{M(t)} \right] + \frac{\Sigma \, \Sigma \, \varphi_i \, \varphi_j \, p_i \, p_j \, \sigma_i \, \sigma_j \, \varrho_{ij}}{\varphi(\underline{p})^2} \Bigg\} dt\,. \end{aligned} \tag{23}$$

For the special case in which $\varphi(\underline{p}) = \prod_{i=1}^{n} p_i^{w_i}$

$$\begin{aligned} E\left(\frac{d\frac{M}{\varphi}}{\frac{M}{\varphi}} \right) = \Bigg\{ & \frac{y(t) - \underline{x}' \, \underline{p}(t)}{M(t)} - \Sigma \, w_i \, \eta_i + \frac{1}{2} \sum_i \sum_j w_i \, w_j \, \sigma_i \, \sigma_j \, \varrho_{ij} + \\ & + \phi'(0) \left[1 - \frac{M^*}{M(t)} \right] \Bigg\} dt \end{aligned} \tag{24}$$

since $\varphi_i = w_i \varphi / p_i$ and $\varphi_{ij} = w_i w_j \varphi / p_i p_j$. For this case, the expected percentage rate of change of the real balances measured by the household varies inversely with a weighted sum of the expected rates of change of the individual prices and directly with a weighted sum of their variances.

For the special case in which $\varphi(\underline{p}) = \sum_i w_i p_i$ for some weights w_i, we have

$$(25) \qquad E\left(\frac{d M / \varphi}{M / \varphi}\right) = \left\{\frac{y(t) - \underline{x}' \underline{p}(t)}{M(t)} - \frac{\sum w_i \eta_i p_i}{\varphi(\underline{p})} + \right.$$

$$\left. + \frac{\sum_i \sum_j w_i w_j p_i p_j \sigma_i \sigma_j \varrho_{ij}}{\varphi(\underline{p})^2} + \phi'(0)\left[1 - \frac{M^*}{M(t)}\right]\right\} dt .$$

The first three terms of (24) and (25) are the counterparts of the familiar equation

$$(26) \qquad \mathrm{dlog}\, M / P = \dot{M} / P - (M / P)(\dot{P} / P) \quad \text{or}$$

i.e., real money balances change passively with the inflow of nominal balances and with the depreciation of real balances by the rate of inflation. The term with the variances arises because of the continuous time stochastic nature of the problem. In addition, real balances can experience a discontinuous jump given by the final term, the relative change in nominal balances caused by a new decision weighted by the likelihood of not making a decision.

$M^*(t)$ is the policy function of the household for nominal money holdings at time t, given that it makes a decision at time t. As such it is dependent on the state of the household, i.e., the household's wealth, money holdings, consumption basket, income, and expectations about the underlying parameters of the economy. In particular, $M^*(t)$ is a function of the expected rate of change of the individual prices, of the variance of those rates of change, and of the levels of the prices. However, as I do not have an explicit solution for $M^*(t)$, I can only make conjectures about its possible form.

4. Comparison to Partial Adjustment Models

It is instructive to relate some parameters and variables of equation (24) to those of a simple, partial adjustment model of money demand. The model used here as an example will be that developed by *Feige* (1967).

Feige's model is intended to identify some structural parameters in a complete system of equations by means of a reduced form estimate of the demand for money. These parameters are the coefficients of expectation of income and of the rate of interest and the partial adjustment parameter of money holdings. Since equation (24) has a term that looks like a partial adjustment term, I concentrate on Feige's model for the partial adjustment of money balances. Specifically, the household is assumed to have a long-run demand for real balances M_t^* which is dependent on y_t^e and r_t^e, the expected real income and the expected interest rate, respectively. The household is assumed to suffer a cost c_1 in increased risk and inconvenience if its money holdings deviate from M_t^* where

$$c_1 = \alpha (M_t - M_t^*)^2 . \tag{27}$$

There is also a transactions cost of adjustment of money balances c_2 where

$$c_2 = \delta (M_t - M_{t-1})^2 . \tag{28}$$

The household minimizes its costs using M_t as a control by setting

$$M_t = \frac{\alpha}{\alpha + \delta} M_t^* + \frac{\delta}{\alpha + \delta} M_{t-1} . \tag{29}$$

If $\gamma \equiv \dfrac{\alpha}{\alpha + \delta}$, then

$$M_t - M_{t-1} = \gamma (M_t^* - M_{t-1}) . \tag{30}$$

The coefficient of adjustment is then a function of the parameters in the total cost function.

A difficulty with this approach is the special nature of the utility function implied by c_1. Also, since M_t is a stock, decisions to change it should affect future levels of utility; and an optimal M_t should result from some more complex dynamic model.

However, leaving these points aside, equation (24) can be compared to (30) by multiplying through by $M(t) / \varphi(t)$ to yield

$$E \frac{dM(t)}{\varphi(t)} = \left\{ \frac{y(t) - \underline{x}' \, \underline{p}(t)}{\varphi(t)} - \frac{M(t)}{\varphi(t)} \left(\sum w_i \eta_i - \frac{1}{2} \sum_i \sum_j w_i w_j \sigma_i \sigma_j \varrho_{ij} \right) + \right.$$
$$\left. + \; \phi'(0) \left[\frac{M(t)}{\underline{\varphi}(t)} - \frac{M^*(t)}{\varphi(t)} \right] \right\} dt . \tag{31}$$

Since $\phi'(0)$ is negative (31) appears to have elements of a partial adjustment model. However, $\psi'(0)$, where $\psi'(0) \equiv -\phi'(0)$, is the value of the p.d.f. of first passage at time t. The functional form of $\psi'(0)$ is determined partly by the household at the time of its last decision, through the choice of $\underline{x}$ and M^*. These, in turn, are based on the future expected paths of decision costs $c(t)$ and on the present value of the utility cost of not consuming the optimal bundle. Therefore, $\psi'(0)$ encompasses costs that are similar in nature to those proposed by Feige. The coefficient $\psi'(0)$ is theoretically more appealing because it is based on a more general form of the underlying utility function than in Feige's formulation, but it may offset this benefit by its complexity.

The term $[M^*(t) / \varphi(t) - M(t)\,\varphi(t)]$ can also be related to the partial adjustment framework. $M^*(t)$ is the nominal balances that the household would hold if a decision were made at time t. However, $M(t)$ is not adjusted purposefully toward $M^*(t)$ when no decision is made. Rather, $M(t)$ changes randomly and independently of $M^*(t)$. When a decision is made, $M(t)$ jumps instantly to $M^*(t)$. This differs from the Feige partial adjustment result because of the lump-sum nature of the decision cost.

The higher the value of $\psi'(0)$, the greater are the chances that $M(t)$ and $M^*(t)$ coincide. If a new dicision is made at time t, i.e., if $\psi'(0) = \infty$, then $M^*(t) = M(t)$, the "desired" cash holdings. This result is identical to the result for a continuous time adjustment process where the coefficient to adjustment is infinite. The difference is that $\psi'(0)$ varies through time and reaches infinity only at discrete intervals because of the lump sum decision cost.

Summary

This paper presents a complex optimization problem in the household's demand for money taking computation costs into account.

The production of a decision is considered an investment and an agent will not plan to replace an old decision until the expected value of a new decision exceeds its expected cost. It is assumed that the household determines the appropriate time to make a new decision according to the level of its money balances. A major result of the model is that the expected time interval between revisions of the household's decisions is one of the variables determining the demand for money.

Zusammenfassung

Diese Abhandlung hat zum Gegenstand die Lösung eines komplexen Optimierungsproblems für die Haushaltsnachfrage nach Geld unter Berücksichtigung der Berechenbarkeitskosten.

Es wird für die mikroökonomische Theorie des Geldes nachgewiesen, daß Nachfrageentscheidungen wie Investitionsentscheidungen ablaufen, wobei eine neue

Entscheidung nur dann eine alte ablöst, wenn der Erwartungswert der neuen Entscheidung ihre erwarteten Kosten übersteigt.

Als wesentliches Ergebnis aus diesem Modell erhalten wir die Aussage, daß das erwartete Zeitintervall für die Revision der Haushaltsentscheidungen ein bestimmender Faktor für die Geldnachfrage ist.

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Appendix

a) *Derivation of Equation (5)*

In the interval $(t, t + k)$ there are three possibilities:

1. The event of first passage does not occur in the interval.
2. The event of first passage occurs once in the interval.
3. The event of first passage occurs more than once in the interval, i.e., more than one decision is made in the interval.

In the first case,

$$\text{(a)} \quad V(t) = U(\underline{x}) \frac{1}{\varrho}(1 - e^{-\varrho k}) + e^{-\varrho k} E_t^* \{ U(\underline{x}) \int_k^{t_1} e^{-\varrho \tau} d\tau + $$

$$+ e^{-\varrho (t_1 - k)} \max_{\underline{x}^*, M^*} V[W(t) - c(t) + M(t) - M^*, \underline{p}(t_1), y(t_1), \underline{x}^*, M^*)\}$$

$$= U(\underline{x}) \frac{1}{\varrho}(1 - e^{-\varrho k}) + e^{-\varrho k} E_t^* V[W(t), \underline{p}(t + k), y(t + k), \underline{x}, M(t + k)]$$

where the expectation operator E_t^* is conditional on the first passage's not having occurred before $t + k$, and $V(t)$ is a shorthand for V evaluated at the time t variable values.

In the second case, assuming the event occurs at $t + \lambda k$ for $0 < \lambda < 1$, then

$$\text{(b)}\qquad V(t) = U(\underline{x}) \left[\frac{1}{\varrho}(1 - e^{-\lambda\varrho k}) \right] + e^{-\varrho\lambda k} \max_{\underline{x}^*,\, M^*} E_t^{**}\, V[w(t) - c(t + \lambda k) +$$

$$+ M(t + \lambda k) - M^*, \underline{p}(t + \lambda k), y(t + \lambda k), \underline{x}^*, M^*]$$

where E_t^{**} is the expectation operator conditional on having to make a decision at $t + \lambda k$.

The third possibility is assumed to have a probability that is $o(k)$ and is therefore ignored. The meaning of the first passage's occurring more than once in the interval should be made clear. If money holdings hit a barrier at time $t + \alpha k$, $0 < \alpha < 1$, the household makes an immediate decision and money holdings jump discontinuously back within their limits. However, a finite period of time $(1 - \alpha)$ remains; and it is possible for the money process to pass through a barrier again in the remaining interval.

The probability of event 1) is $\phi(k)$. That for event 2) is approximately $[1 - \phi(\mathrm{k})]$. The expected value of $V(t)$ is then

$$\text{(c)}\qquad V(t) = \phi(k) \Big\{ U(\underline{x}) \frac{1}{\varrho}(1 - e^{-\varrho k}) +$$

$$+ e^{-\varrho k} E_t^*\, V[W(t) - \underline{p}(t + k), y(t + k), \underline{x}, M(t + k)] \Big\}$$

$$+ [1 - \phi(k)] \Big\{ U(\underline{x}) \frac{1}{\varrho}(1 - e^{-\varrho\lambda k}) +$$

$$+ e^{-\varrho\lambda k} E_t^{**} \max_{\underline{x}^*,\, M^*} V[W(t) - c(t + \lambda k) + M(t + \lambda k) -$$

$$- M^*, \underline{p}(t + \lambda k), y(t + \lambda k), \underline{x}^*, M^*] \Big\} + o(k)$$

as promised.

b) *Derivation of Equation (7)*

In equation (6),

$$\phi(0) = 1$$

$$z_1(0) = V[W(t), y(t), p(t), \underline{x}, M(t)] \equiv V(t)$$

$$z_1'(0) = U(\underline{x}) - \varrho V(t) + V_y E_t^* \Delta y + \sum_{j=1}^{n} V_{P_i} E_t^* \Delta p_i + V_M E_t^* \Delta M +$$

$$+ \frac{1}{2}[E_t^* \Delta \underline{p}' V_{pp} \Delta \underline{p} + E_t^* \Delta \underline{p}^* V_{\underline{p}y} \Delta y + E_t^* V_{yy} \Delta y^2]$$

and $[1 - \phi(0)] = 0$, so the terms $[1 - \phi(0)]\, z_2(0)$ and $[1 - \phi(0)]\, z_2'(0)$ fall out. The term $- \phi(0)]\, z_2(0)$ remains where $z_2(0) = E_t^{**} \max_{\underline{x}^*, M^*} V\,[W(t) - c(t) + M(t) - M^*, y(t), \underline{p}(t), \underline{x}^*, M^*]$. Substituting these results into (6) and subtracting $V(t)$ from both sides yields (7).

c) *Derivation of Equation (19)*

Equation (19) can be derived from equation (18) by expanding (18) about $k = 0$. Carrying out this expansion,

$$
\begin{aligned}
\Delta \frac{M(t)}{\varphi[\underline{p}(t)]} = {} & \phi(0) \frac{M(t)}{\varphi[\underline{p}(t)]} + [1 - \phi(0)] \frac{M^*}{\varphi(\underline{p})} - \frac{M(t)}{\varphi[\underline{p}(t)]} + \\
& + \phi'(0) \frac{M(t)}{\varphi[\underline{p}(t)]} k + \phi(0) \left[\frac{y(t) - \underline{p}'(t)\, \underline{x}}{\varphi[\underline{p}(t)]} k - \frac{M(t)}{\varphi(\underline{p})^2} \Delta \varphi(\underline{p}) \right] + \\
& + \phi(0) \frac{M(t)}{\varphi(\underline{p})^3} (\Delta \varphi)^2 - \phi'(0) \frac{M^*}{\varphi(\underline{p})} k + \\
& + [1 - \phi(0)] \left[\frac{y(t) - \underline{p}'(t)\, \underline{x}}{\varphi(\underline{p})} k - \frac{M^*}{\varphi(\underline{p})^2} \Delta \varphi[\underline{p}(t)] \right] + o(k)\,.
\end{aligned}
$$

The term in $(\Delta \varphi)^2 \dfrac{M(t)}{\varphi(\underline{p})^3}$ is carried along because terms in $\Delta p_i\, \Delta p_j$ are of order k. But since $\phi(0) = 1$, $[1 - \phi(0)] = 0$ and equation (19) follows.