Stock-Dependent Extraction Costs and the Technological Efficiency of Resource Depletion

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A great number of studies on optimal resource extraction in the presence of extraction costs have been carried out, and there also exist some studies where the realistic assumption of stock-dependent extraction costs is made. However, the literature is not very explicit about pure efficiency conditions in the Pareto sense, i.e., conditions that are independent of special assumptions about intertemporal preferences and market structures. The present paper addresses the efficiency problem explicitly and, in particular, tries to remove some confusion remaining in a recent paper by *Heal*.

1. Introduction

In his lecture given to the 1979 conference of the Verein für Socialpolitik[1], *Geoffrey Heal* presented an efficiency condition for intertemporal resource extraction in the presence of extraction costs. This paper illustrates that *Heal*'s condition is fallacious and corrects the mistake. In addition, it demonstrates the compatibility between the corrected condition and the optimality conditions derived in Rawlsian and utilitarian frameworks by *Solow/Wan* (1976) and *Heal* (1976), respectively.

2. Heal's Efficiency Condition

Consider an economy producing a single composite commodity. At each point in time t output Y is given by

$$Y = G(K, R, t) \; ; \qquad G_K, G_R > 0 \; , \qquad G_{KK}, G_{RR} < 0 \; , \tag{1}$$

and resource extraction cost F in terms of the composite commodity is[2]

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[1] *Heal* (1980).

[2] Unlike (2), *Heal* assumes a marginal extraction cost function $C(S, R)$. I refer to a total cost function because this seems more systematic and avoid the

(2)
$$F = X(S, R) \ , \quad X_S < 0 \ , \quad X_R > 0 \ , \quad X_{RR} \geq 0 \ , \quad X_{RS} < 0 \ ,$$

where K is the capital stock, R the rate of resource extraction and S the stock of the depletable resource.

Output is completely used for consumption C, investment in the capital stock I and extraction cost F:

(3)
$$Y = C + I + F \ .$$

The rate of change over time[3] in the capital stock is

(4)
$$\dot{K} = I$$

and the rate of change in the stock of the depletable resource is

(5)
$$\dot{S} = -R \ .$$

Together with the initial stocks K_0 and S_0 (K_0, $S_0 > 0$) the time paths of I and R completely determine the intertemporal allocation pattern in the economy.

The question is under which conditions these time paths constitute an intertemporally efficient allocation of resources. The allocation is said to be efficient if there is no time interval where it is possible to increase consumption without decreasing it at the same time in another interval. Of course one could bypass the efficiency question by adding to the above formulas a well specified welfare functional and calculating the optimal allocation explicity. But in view of the difficulties of intergenerational welfare comparisons it seems useful to separate efficiency and distribution problems analogously to the procedure in static allocation theory.

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symbol C because it is also used for consumption. But of course there does not remain a substantial difference if we define $X(S, R) \equiv \int_0^R C(S, u)\, du$. In light of the time dependence of the production function it would seem more systematic to use an extraction cost function $X(S, R, t)$. But this version would not be compatible with *Heal*. Our results would, however, still go through.

Perhaps the function $X(S, R)$ should be called a factor-input function rather than a cost function since there are no prices involved. On the other hand, with output Y as the numéraire, the price by which we had to multiply in order to have a cost function is unity.

$X_S < 0$ reflects the assumption that the order of extraction from different deposits is such that the lowest cost deposit is extracted first, the others following in strict sequence. This assumption by itself can be shown to follow from the requirement of efficiency if the rate of return on capital is strictly positive and if the extraction industry uses a part of output Y as an input. Cf. *Kemp/Long* (1980 a, b, d) and *Sinn* (1981).

[3] We define $\dot{Z} = dZ/dt$ where t is a time index.

Among the conditions necessary for efficiency we should particularly be interested in a marginal condition relating to each other the effects of capital investment and resource extraction. Under the absence of extraction costs *Heal* (p. 42 - 44) shows that the growth rate of the marginal product of the resource should be equal to the rate of return on capital:

$$G_K = \frac{d}{dt} \ln G_R \ . \tag{6}$$

This condition has also been derived by *Solow* (1974) and *Stiglitz* (1974) in Rawlsian and utilitarian frameworks, and in a competitive economy it would automatically be satisfied since it would then be the same as the Hotelling rule with G_K as the market rate of interest and G_R as the market price of the resource.

It is not surprising that (6) does not hold any more if there are extraction costs. Without proof *Heal* claims (pp. 46, 48) for this case that the net marginal product of the resource, i.e., its marginal product minus its marginal extraction cost, should change at a rate given by the rate of return on capital:

$$G_K = \frac{d}{dt} \ln (G_R - X_R) \ . \tag{7}$$

Furthermore he stresses (pp. 46, 80) that (7) implies that the growth rate of the marginal productivity of the resource will be equal to a weighted average of the return on capital, G_K, and the rate of change of marginal extraction costs,

$$\frac{\dot{G}_R}{G_R} = G_K \left(1 - \frac{X_R}{G_R}\right) + \frac{\dot{X}_R}{X_R} \frac{X_R}{G_R} \ , \tag{8}$$

where the change in marginal extraction costs can itself be explained by a change in the rate of extraction and the stock of the resource[4]:

$$\frac{\dot{X}_R}{X_R} = X_{RR} \frac{R}{X_R} \frac{\dot{R}}{R} - X_{RS} \frac{S}{X_R} \frac{R}{S} \ . \tag{9}$$

Although these efficiency conditions might have some intuitive appeal at first glance, they look suspicious if contrasted with a well-known extension of the Hotelling rule for the case of a competitive market with positive extraction costs[5]:

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[4] In *Heal's* paper this equation shows various typing errors.

[5] See *Levhari/Liviatan* (1977) equ. (15), *Dasgupta/Heal* (1979) p. 169 and *Kemp/Long* (1980 c) equ. (15 b). Cf. also *Pindyck* (1978 a and b). In (1978 b) *Pindyck* allows for exploration costs in addition to extraction costs, but in

(10) $$r = \frac{\dot{p}}{p} - \frac{X_S}{p} \ .$$

Here r denotes the market rate of interest and p the market price of the harvested resource net of marginal extraction costs which under competitive conditions equals the net marginal productivity of the resource, $G_R - X_R$.

Obviously condition (10) is only compatible with (7) if $X_s = 0$, i.e., if, in contrast to *Heal*'s assumption, extraction costs do not depend on the remaining stock of the resource[6]. Thus, if (7) were really true, the competitive economy would not ensure the intertemporal efficiency of resource allocation. In the light of the fundamental theorem of static welfare theory this appears to be a rather strange implication. And in fact conditions (7) and consequently (8) are wrong. It can easily be shown that pure technological efficiency considerations require an extension of these conditions in a way to make them compatible with (10).

3. Derivation of the Correct Condition

The proof is similar to *Heal*'s proof of (6). Starting with a given time path of the economy we conduct marginal variations in the control variables within a limited time span without however changing the intertemporal allocation elsewhere. We can then easily see which conditions have to be satisfied such that no Pareto improvement is possible.

We assume for a while that the economy operates in discrete time with periods of length Θ, $\Theta > 0$, while the reference period for defining the flow variables is unity. A point in time and the subsequent time interval belong together, such that they can be named by the same time index. As the time span for conducting the variations we choose the periods t and $t + \Theta$ and assume that

(11) $$dC_t = dK_t = dK_{t+2\Theta} = dS_t = dS_{t+2\Theta} = 0 \ .$$

The reason for $dC_t = 0$ is that we want to examine whether an increase in second-period consumption is possible without changing consumption

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both papers he excludes the possibility that marginal extraction costs depend on the speed of extraction, R. The connection between (10) and his result can easily be understood after having read this paper and comparing our equation (21) with equation (A.9) in (1978 a) and (9) in (1978 b) for the case of zero exploration costs.

[6] Cf. *Weinstein* and *Zeckhauser* (1975) esp. pp. 379 - 381. These authors derive equation (7) under the assumption that extraction costs depend only on the rate of extraction alternately for the case of a market equilibrium and a utilitarian planning problem.

in the first period. If it *is* possible for a given intertemporal allocation, then this allocation is inefficient. Constant stocks at the beginning and the end of the two-period interval are required by our assumption that the allocation is to be unchanged outside this interval. Since formulas (1) - (3) imply

$$[(G(K_\tau, R_\tau, \tau) - X(S_\tau, R_\tau) - I_\tau - C_\tau] = 0 \,, \qquad \tau = t \,,\ t + \Theta \,, \tag{12}$$

the following two equations have to be satisfied for the variations carried out:

$$dI_t = (G_{R_t} - X_{R_t})\, dR_t \tag{13}$$

$$dC_{t+\Theta} = G_{K_{t+\Theta}} dK_{t+\Theta} - dI_{t+\Theta} - X_{S_{t+\Theta}} dS_{t+\Theta} + (G_{R_{t+\Theta}} - X_{R_{t+\Theta}})\, dR_{t+\Theta} \,. \tag{14}$$

Now, $K_{t+\Theta} = K_t + I_t \Theta$ and $S_{t+\Theta} = S_t - R_t \Theta$; hence $dK_{t+\Theta} = dI_t \Theta$ and $dS_{t+\Theta} = -dR_t \Theta$. Furthermore (11) implies that $dI_t = -dI_{t+\Theta}$ and $dR_{t+\Theta} = -dR_t$. Thus, (13) and (14) can be combined to

$$dC_{t+\Theta} = [(1 + \Theta\, G_{K_{t+\Theta}})(G_{R_t} - X_{R_t}) + \Theta\, X_{S_{t+\Theta}} - (G_{R_{t+\Theta}} - X_{R_{t+\Theta}})]\, dR_t \,. \tag{15}$$

This formula shows by how much consumption in the second period can rise, if capital investment and resource extraction are increased in a way that keeps first-period consumption unchanged.

Obviously, if the variation is conducted around an efficient time path, we have $dC_t = 0$ by definition. Hence it is readily apparent from path, we have $dC_{t+\Theta} = 0$ by definition. Hence it is readily apparent from (15) that

$$G_{K_{t+\Theta}} = \frac{(G_{R_{t+\Theta}} - X_{R_{t+\Theta}}) - (G_{R_t} - X_{R_t})}{(G_{R_t} - X_{R_t})\,\Theta} - \frac{X_{S_{t+\Theta}}}{G_{R_t} - X_{R_t}} \tag{16}$$

is a necessary condition for an efficient intertemporal allocation.

So far, the argument has been carried out for $\Theta > 0$. But by choosing Θ sufficiently small we can approach the continuous case as closely as we wish. Accordingly the condition can then also be written as

$$G_K = \frac{dt}{d} \ln (G_R - X_R) - \frac{X_S}{G_R - X_R} \tag{17}$$

or, equivalenty, as

$$\frac{\dot{G}_R}{G_R} = G_K \left(1 - \frac{X_R}{G_R}\right) + \frac{\dot{X}_R}{X_R} \frac{X_R}{G_R} + \frac{X_S}{G_R} \,, \tag{18}$$

where all variables refer to the same point in time. Equation (17) is, as we expected, indeed analogous to the competitive condition (10) proving the intertemporal efficiency of the competitive market allocation. Since extraction costs rise with a fall in the stock of the resource, $X_S < 0$, (17) implies that, unlike *Heal*'s contention [equ. (7)], *the net marginal product of the resource should change by a rate less than the rate of return on capital.* In addition, a comparison between (8) and (18) shows that *the growth rate of the gross marginal productivity of the resource is not just a weighted average* of the rate of return on capital and the growth rate of marginal extraction costs, *but smaller than this* by X_S/G_R.

An intuitive explanation of our result can be given as follows: There are two tools for shifting consumption from the first period to the second. The first is an increase in investment. If one unit of consumption is substituted by capital investment, then second-period consumption can be increased by one unit plus the rate of return of return on capital. The second tool is a reduction in the rate of resource extraction. Suppose resource extraction in the first period falls by an amount given by the reciprocal value of the net marginal productivity of the resource, such that consumption in this period is reduced by a unit. Then, second-period consumption can be increased by a unit plus the percentage increase in the net marginal productivity plus, and this is the new element, the decrease in second-period extraction costs effected by the availability of a higher resource stock[7]. If the intertemporal allocation is to be Pareto optimal then the possible increase in second-period consumption must be the same for each tool, for only then it is impossible to alter both resource extraction and investment in a way that keeps first-period consumption constant, but increases consumption in the second period.

[7] To provide further intuition for the result, suppose, before the variation is conducted, there is a constant unit extraction cost within each of the two periods considered that depends only the resource stock available at the beginning of the corresponding period. Then a resource unit the extraction of which is shifted from the first period to the second can be extracted at a cost that is below the previous unit (and marginal) extraction cost in the second period by the amount $-X_{S_{t+\Theta}}$. Hence, if the extraction of $1/(G_{R_t} - X_{R_t})$ of the resource is shifted from the first period to the second, the increase in extraction costs in the second period would be overestimated by $-X_{S_{t+\Theta}}/(G_{R_t} - X_{R_t})$ if it would be taken to be $1/(G_{R_t} - X_{R_t})$ times the unit extraction cost in the second period before the variation.

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Equations (17) and (18) have been derived for a very general extraction cost function. It is however worth to consider the special, although still plausible, case[8]

$$X(S, R) = R\,g(S)\ ,\quad g' < 0\ , \tag{19}$$

where unit extraction costs g depend only on the remaining stock of the resource. Since the simplified extraction cost function implies that $X_R = g(S)$, $\dot{X}_R = g'\dot{S} = -g'R$ and $X_S = Rg'$, (17) and (18) can be reduced to

$$G_K = \frac{\dot{G}_R}{G_R - X_R} \tag{20}$$

and

$$\frac{\dot{G}_R}{G_R} = G_K\left(1 - \frac{X_R}{G_R}\right)\ . \tag{21}$$

in this case. *Hence, the absolute rate of change of the gross marginal productivity of the resource relative to the net marginal productivity equals the rate of return of capital, and the relative rate of change of the gross marginal productivity is a share of the rate of return on capital where the share is given by the ratio of net to gross marginal productivity.* The reader should contrast (20) and (21) with *Heal*'s equations (7) and (8).

4. Comparison with the Utilitarian Optimum

As support of the fallacious equation (8) *Heal* cites his 1976 paper in the Bell Journal[9]. The paper does not directly address the efficiency problem since the analysis is carried out in a utilitarian framework. But efficiency is a necessary condition for a utilitarian optimum. Thus we should elaborate briefly upon the relationship to our results.

The problem studied in the Bell paper is to find optimality conditions under the utilitarian aim

$$\max \int_0^\infty u(C_t)\, e^{-\delta t}\, dt \tag{22}$$

[8] A function of this type has frequently been used. Perhaps the first promoter was *Gordon* (1954). Recent examples are *Heal* (1976), *Pindyck* (1978 a and b) and *Solow/Wan* (1976).

[9] *Heal* (1980), 80: "I ... can only mention briefly that the characterization I gave of efficient price paths — price changes equal to a weighted average of interest rates and marginal cost changes — can also be shown to hold for a resource available in a range of deposits of different qualities. This is shown in an article of myself in the Bell Journal 1976." Cf. *Heal* (1976).

where u is a strictly concave utility function and δ the rate of time preference. Otherwise the model is (in the relevant aspects) the same as here. The extraction cost function is of type (19). *Heal* shows that the solution of this problem indeed provides an optimality condition somewhat similar to (8):

$$\frac{\dot{\tilde{p}}}{\tilde{p}} = \delta\left(1 - \frac{\tilde{c}}{\tilde{p}}\right) + \frac{\dot{u}'}{u'}\,\frac{\tilde{c}}{\tilde{p}}\,, \tag{23}$$

$$\tilde{p} = u'G_R\,,\quad \tilde{c} = u'g\,.$$

Here the growth rate of the marginal value product is shown to be a weighted average of the discount rate and the relative change of the output price in utility terms, where the weights are the same as those in (8). The formal similarity is, however, meaningless.

Note that for a Hamiltonian of the kind $H = e^{-\delta t}\{u(C) + p\,[G(K, R) - g(S)\,R - C] + q(-R)\}$ the equation $\frac{\dot{u}'}{u'} + G_K = \delta$ is a necessary condition for an interiour optimum. Thus (23) can be written as

$$\frac{\dot{G}_R}{G_R} = \delta\left(1 - \frac{\tilde{c}}{\tilde{p}}\right) - \frac{\dot{u}'}{u'}\left(1 - \frac{\tilde{c}}{\tilde{p}}\right) = G_K\left(1 - \frac{X_R}{G_R}\right), \tag{24}$$

which is the same as (21).

5. Comparison with the Rawlsian Optimum

Another study in resource extraction with stock-dependent extraction costs is that of *Solow* and *Wan* (1976). Inspired by Rawls' minimax rule these authors examine the conditions for maximizing the level of a steady, time-invariant flow of consumption. Since technological efficiency is a necessary condition for a Rawlsian optimum, we again should be able to demonstrate the compatibility with our results.

Although formally somewhat different, the technological assumptions of *Solow* and *Wan* are those of this paper with $X(S, R) = R\,g(S)$[10]. Their approach can be stated as follows. The dual problem of maximizing consumption for a given stock of the resource is to minimize accumulated resource extraction for a given level of consumption $\bar{C}$. Hence we

$$\max_{R} \int_0^\infty -R(t)\,dt \tag{25}$$

s.t.

[10] Cf. *Solow/Wan* (1976) fn. 3.

$$\dot{K} = G(K, R) - R\,g(S) - \bar{C}\ ,$$
$$\dot{S} = -R\ ,$$

given the initial stocks K_0 and S_0. The Hamiltonian for this problem is

$$H = -R + p^* [G(K, R) - R\,g(S) - \bar{C}] + q^* (-R)\ .$$

From $\partial H / \partial R = 0$ we achieve[11]:

$$G_R - g(S) = \frac{1 + q^*}{p^*}\ , \tag{26}$$

from $p^* = -\partial H / \partial K$:

$$-\frac{\dot{p}^*}{p^*} = G_K\ , \tag{27}$$

and from $q^* = -\partial H / \partial S$:

$$\dot{q}^* = p^* R\, g'(S)\ . \tag{28}$$

Solow and *Wan* do not derive anything out of these conditions that resembles one of the various versions of our efficiency condition. Nevertheless it is straightforward to do this. (26) implies that

$$\frac{d}{dt}\, ln\, [G_R - g(S)] = \frac{\dot{q}^*}{1 + q^*} - \frac{\dot{p}^*}{p^*}\ . \tag{29}$$

Inserting (27) and (28) we can write this equation in the form

$$\frac{d}{dt}\, ln\, [G_R - g(S)] = \frac{p^*}{1 + q^*} R\, g'(S) + G_K\ . \tag{30}$$

Because of (26) we then have

$$G_K = \frac{d}{dt}\, [ln\, G_R - g(S)] - \frac{R\, g'(S)}{G_R - g(S)}\ , \tag{31}$$

which is our equation (17) for $X(S, R) = R\, g(S)$.

[11] The equivalence between conditions (26) - (28) and conditions (8) - (10) in the *Solow/Wan* paper becomes obvious with the following equalities. $p^* = p$, $q^* = q$, $G(K, R) = K^a R^b$, $g[S(t)] = \Theta(t)$. Differentiation of the latter condition yields $g' = \dot{\Theta}/\dot{S} = -\dot{\Theta}/R$. Together with equation (7) from *Solow/Wan* this implies $g'(S) = -1/f(\Theta)$.

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Summary

The paper deals with purely technological efficiency conditions for resource extraction in the presence of stock-dependent extraction costs. While *Heal* contended the marginal productivity of the resource had to grow at a rate given by a weighted average of the rate of return on capital and the time change of marginal extraction costs, it is demonstrated that a different condition must hold which requires a lower rate of growth. The result is shown to be compatible with optimality conditions that have been derived from Rawlsian and utilitarian planning problems.

Zusammenfassung

Der Aufsatz behandelt rein technologische Effizienzbedingungen der Ressourcenextraktion bei bestandsabhängigen Extraktionskosten. Während *Heal* behauptet hat, die Grenzproduktivität der Ressource müsse mit einer Rate wachsen, die einem gewogenen Mittel der Grenzproduktivität des Kapitals und der Wachstumsrate der marginalen Extraktionskosten entspreche, wird hier gezeigt, daß eine andere Bedingung zu gelten hat, die ein geringeres Wachstum verlangt. Es wird nachgewiesen, daß das Ergebnis mit Optimierungsbedingungen vereinbar ist, die bereits aus Rawlsianischen und utilitaristischen Planungsproblemen abgeleitet worden sind.

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