The Structure of Technology and Tests of a Model of Production: Reply and Further Results

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The critical comment by *Zweifel* (this issue) has been the motivation to carry out once more the tests of hypotheses from production theory by allowing also non-zero profits. We furthermore consider implicite and non-homogeneous production functions and repeat the tests to show the sensitivity of the results with respect to more general models. The sensitivity of the results underlines that there is no way to test the theory of production as such. What can be done is only to test a specific model of production under the assumption made in developing the analytical framework.

1. Introduction

In the paper commented by *Zweifel* (this issue) *Conrad* and *Jorgenson* (1978) (*CJ* henceforth) employed econometric models of production to develop tests of parametric restrictions on pattern of substitution and technical change implied by separability in commodities and time. *Zweifel* noticed correctly that the results of the tests are based on the assumption that an implicit transformation function can be written in an explicit form. There is nothing wrong with that as empirical results have to be interpreted always in the light of the assumption made. If one has serious doubts with respect to the assumptions, one might cast doubt on the validity of the empirical results. To achieve greater generality, one can start, of course, with an implicit transformation function but this might also not increase the readers' confidence in the empirical results if he believes that the hypothesis of profit maximization under given prices is unrealistic in a world of monopolistic behavior. It might be interesting to show the sensitivity of the results by employing the implicit production function. In this case, however, there is more than one production function which can generate translog value shares. As there corresponds to each production function a different set of parameter restrictions, more than one test is required for testing the hypothesis of profit maximization.

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The hypothesis of profit maximization under the assumption of zero profits and an explicit production function has been tested and not rejected in *CJ* (1977). We do not have to test for a change of functional forms of the share equations as commented by *Zweifel*, as the translog representation of marginal productivity and supply and demand functions can be interpreted as a first-order approximation of any system of marginal productivity and supply and demand functions. In section 2 of this paper we will develop a model of production based on the assumption of non-zero profits and an explicitly written production function. Results of the test of the hypothesis of profit maximization are given in section 3. In section 5 we develop a more general model of production based on an implicit production function, and the empirical results show that more work has to be done to carry out a conclusive test procedure.

2. Production model I: Separable, Homogeneous production function; one input fixed

As in *CJ* (1977, 1978), we specialize to the case of two outputs — consumption C and investment I — and two inputs — labor L and capital K. The corresponding prices are q_C, q_I, q_L and q_K, respectively, and t is time, considered an index of technology. The objective of the firm is to maximize short-run profits subject to a given labor input[1]:

$$\text{(1)} \qquad \underset{C, I, K}{\text{Max}} \quad \{\pi = q_C C + q_I I - q_K K - q_L L \,|\, L = F(C, I, -K, t)^r\} \, .$$

The production function is assumed to be homogeneous of degree r, that is, F is homogeneous of degree one.

We rewrite the production function as $L^{1/r} = F(\cdot)$ and obtain the necessary conditions for producer equilibrium as follows:

$$\text{(2)} \qquad q_C = \lambda F_C \; , \quad q_I = \lambda F_I \; , \quad -q_K = \lambda F_K \; ,$$

where λ is the Lagrange multiplier. From Euler's theorem we obtain

$$q_C C + q_I I - q_K K = \lambda [F_C C + F_I I + F_K \cdot K] = \lambda L^{1/r} \; ,$$

or

$$\lambda = \frac{\pi + q_L L}{L^{1/r}} \; .$$

[1] It is more realistic to treat capital rather than labor as fixed in a short-run model of production. But in this case the share equations are different from the ones estimated in *CJ* (1977, 1978) and we can not compare the results of the tests.

Substituting this expression for λ into (2) results in necessary conditions for producer equilibrium saying that elasticities of the production function F with respect to consumption, investment and capital are equal to the value shares of goods to the sum of profits and wages:

$$w_C := \frac{q_C C}{\pi + q_L L} = \frac{\partial \ln F}{\partial \ln C}\ , \quad w_I := \frac{q_I I}{\pi + q_L L} = \frac{\partial \ln F}{\partial \ln I}\ , \qquad w_K := -\frac{q_K K}{\pi + q_L L} = \frac{\partial \ln F}{\partial \ln K} \tag{3}$$

For specifying the production function we can either approximate the production function by a Taylor's series and derive (3), or we can approximate the shares in (3) and solve the system of partial differential equations to obtain the production function. We choose the translog representation of the production function F

$$\begin{aligned} L^{1/r} = exp\ [&\alpha_0 + \alpha_C \ln C + \alpha_I \ln I + \alpha_K \ln K + \alpha_t \cdot t \\ &+ {}^1/_2 \{\beta_{CC} (\ln C)^2 + \beta_{CI} \ln C \ln I + \beta_{CK} \ln C \ln K \\ &+ \beta_{IC} \ln I \ln C + \beta_{II} (\ln I)^2 + \beta_{IK} \ln I \ln K \\ &+ \beta_{KC} \ln K \ln C + \beta_{KI} \ln K \ln I + \beta_{KK} (\ln K)^2\} \\ &+ \beta_{Ct} \ln C \cdot t + \beta_{It} \ln I \cdot t + \beta_{Kt} \ln K \cdot t + {}^1/_2 \beta_{tt} \cdot t^2]\ , \end{aligned} \tag{4}$$

and as F is assumed to be homogeneous of degree one, the usual parameter restrictions must be satisfied:

$$\begin{aligned} &\alpha_C + \alpha_I + \alpha_K = 1\ , \\ &\beta_{CC} + \beta_{IC} + \beta_{KC} = 0\ , \\ &\beta_{CI} + \beta_{II} + \beta_{KI} = 0\ , \\ &\beta_{CK} + \beta_{IK} + \beta_{KK} = 0\ , \\ &\beta_{Ct} + \beta_{It} + \beta_{Kt} = 0\ . \end{aligned} \tag{5}$$

The equations for the shares follow from (3) and (4):

$$\begin{aligned} w_C &= \alpha_C + \beta_{CC} \ln C + \beta_{CI} \ln I + \beta_{CK} \ln K + \beta_{Ct} \cdot t \\ w_I &= \alpha_I + \beta_{IC} \ln C + \beta_{II} \ln I + \beta_{IK} \ln K + \beta_{It} \cdot t\ , \\ w_K &= \alpha_K + \beta_{KC} \ln C + \beta_{KI} \ln I + \beta_{KK} \ln K + \beta_{Kt} \cdot t\ , \end{aligned} \tag{6}$$

Next, we can differentiate the production function F logarithmically with respect to time to obtain the rate of technical change

$$w_t := \frac{\partial \ln F}{\partial t} = \frac{1}{r} \frac{\partial \ln L}{\partial t} \tag{7}$$

The rate of technical change is the decline of labor input with respect to time, holding outputs and capital input constant, corrected by the influence of returns to scale. Under the translog specification we obtain

$$w_t = \alpha_t + \beta_{tC} \ln C + \beta_{tI} \ln I + \beta_{tK} \ln K + \beta_{tt} \cdot t \ . \tag{8}$$

To measure the rate of technical change w_t, we differentiate the logarithm of the production function F totally with respect to time

$$\frac{1}{r} \dot{L} = \frac{\partial \ln F}{\partial t} + \frac{\partial \ln F}{\partial \ln C} \dot{C} + \frac{\partial \ln F}{\partial \ln I} \dot{I} + \frac{\partial \ln F}{\partial \ln K} \dot{K}$$

$$= \frac{\partial \ln F}{\partial t} + w_C \dot{C} + w_I \dot{I} + w_K \dot{K}$$

$$\text{where} \quad \dot{C} := \frac{d \ln C}{dt} \text{ etc.}$$

From this expression we get the rate of technical change w_t in terms of the value shares, the rate of growth of the inputs and outputs and of the degree of homogeneity:

$$w_t = \frac{1}{r} \dot{L} - w_C \dot{C} - w_I \dot{I} - w_K \dot{K} \tag{9}$$

We proceeded by deriving the system of marginal productivity functions (6) and (8) given the translog approximation of the production function. As the order of the second derivatives of F can be changed, the matrix (β_{ij}) is symmetric. An alternative procedure is to approximate the marginal productivity conditions in (3) by a Taylor's series of the first order in the logarithms of the variables. As the shares sum to unity, the parameter restrictions in (5) must be satisfied. Furthermore, we observe:

$$\frac{\partial w_C}{\partial \ln I} = \frac{\partial^2 \ln F}{\partial \ln I \, \partial \ln C} \text{ and } \frac{\partial w_I}{\partial \ln C} = \frac{\partial^2 \ln F}{\partial \ln C \, \partial \ln I}$$

which implies the symmetry conditions as necessary and sufficient for the marginal productivity functions to be generated from a translog production function:

$$\begin{aligned} &\beta_{IC} = \beta_{CI} \ , \quad \beta_{CK} = \beta_{KC} \ . \quad \beta_{IK} = \beta_{KI} \\ &\beta_{Ct} = \beta_{tC} \quad \beta_{It} = \beta_{tI} \quad \beta_{Kt} = \beta_{tK} \end{aligned} \tag{10}$$

The hypothesis of the equality of the marginal rate of substitution and transformation with the relative prices is equivalent to the hypothesis of profit-maximizing supply and demand functions. An alternative way of testing this hypothesis is the dual approach to production theory by characterizing the production function in terms of the profit function. To derive the corresponding profit function, we observe that the first-order partial derivatives of F are homogeneous of degree zero; so we rewrite (2) as follows:

$$q_C = \lambda F_C\left(\frac{C}{L^{1/r}}, \frac{I}{L^{1/r}}, -\frac{K}{L^{1/r}}, t\right),\ q_I = \lambda F_I(\cdot)\ ,\quad -q_K = \lambda F_K(\cdot)$$

and $\quad 1 = F\left(\frac{C}{L^{1/r}}, \frac{I}{L^{1/r}}, -\frac{K}{L^{1/r}}, t\right),$

and solve this system for the unknown quantitites in terms of the prices q_C, q_I and q_K:

$$\frac{C}{L^{1/r}} = f_1(q_C\,,\ q_I\,,\ q_K\,,\ t)$$

and similary for $\frac{I}{L^{1/r}}$ and $\frac{K}{L^{1/r}}$

Inserting the optimal production plan $C = L^{1/r} f_1(\cdot), \ldots,$ into the objective function (1), we obtain the profit function $\pi(q_C, q_I, q_K, t)$ in the following partitioning:

$$\pi(q_C\,,\ q_I\,,\ q_K\,,\ t) = L^{1/r} P(q_C\,,\ q_I\,,\ q_K\,,\ t) - q_L L\ . \tag{11}$$

If the profit function is identically zero, as assumed in *CJ* (1977, 1978) then the production function is homogeneous of degree one and (11) implies the price function

$$q_L = P(q_C\,,\ q_I\,,\ q_K\,,\ t)$$

employed in *CJ* (1977, 78). Here we do not assume that the profit function is identically zero.

According to *McFadden*[2] we obtain the profit-maximizing production plan by the partial derivative of the profit function with respect to the prices:

$$\frac{\partial \pi(\cdot)}{\partial q_C} = L^{1/r}\ \frac{\partial P(\cdot)}{\partial q_C} = C$$

or

$$\frac{\partial \ln P(\cdot)}{\partial \ln q_C} = \frac{q_C C}{L^{1/r} P(\cdot)} \text{ (etc. for } I \text{ and } K)\ .$$

Because of (11), this expression can be rewritten as follows:

$$w_C = \frac{q_C C}{\pi + q_L L} = \frac{\partial \ln P(\cdot)}{\partial \ln q_C}\,. \tag{12}$$

Similarly, we obtain:

$$w_I = \frac{q_I I}{\pi + q_L L} = \frac{\partial \ln P(\cdot)}{\partial \ln q_I}\,, \tag{13}$$

[2] See *Fuss* and *McFadden* (1979).

$$w_K = -\frac{q_K K}{\pi + q_L L} = \frac{\partial \ln P(\cdot)}{\partial \ln q_K}. \tag{14}$$

We next specify the price function $P(\cdot)$ as a translog price function:

$$\begin{aligned} P(\cdot) = exp\,[&\alpha_0 + \alpha_C \ln q_C + \alpha_I \ln q_I + \alpha_K \ln q_K + \alpha_t \cdot t \\ &+ {}^1\!/\!_2 \{\beta_{CC} (\ln\ q_C)^2 + \beta_{CI} \ln q_C \ln q_I + \beta_{CK} \ln q_C \ln q_K \\ &+ \beta_{IC} \ln q_I \ln q_C + \beta_{II} (\ln q_I)^2 + \beta_{IK} \ln q_I \ln q_K \\ &+ \beta_{KC} \ln q_K \ln q_C + \beta_{KI} \ln q_K \ln q_I \\ &+ \beta_{KK} (\ln q_K)^2] \\ &+ \beta_{Ct} \ln q_C \cdot t + \beta_{It} \ln q_I \cdot t + \beta_{Kt} \ln q_K \cdot + {}^1\!/\!_2\, \beta_{tt} \cdot t^2]\ . \end{aligned} \tag{15}$$

As the profit function is homogeneous of degree one in the prices, the price function $P(\cdot)$ is homogeneous of degree one in the prices q_C, q_I and q_K. This implies that the parameters of the translog price function satisfy restrictions that are precisely analogous to the restrictions on the parameters of the translog production function given under (5).

By logarithmic differentiation of the translog representation of the price function (15) with respect to prices we obtain:

$$\begin{aligned} w_C &= \alpha_C + \beta_{CC} \ln q_C + \beta_{CI} \ln q_I + \beta_{CK} \ln q_K + \beta_{Ct} \cdot t\ , \\ w_I &= \alpha_I + \beta_{IC} \ln q_C + \beta_{II} \ln q_I + \beta_{IK} \ln q_K + \beta_{It} \cdot t\ , \\ w_K &= \alpha_K + \beta_{KC} \ln q_C + \beta_{KI} \ln q_I + \beta_{KK} \ln q_K + \beta_{Kt} \cdot t\ . \end{aligned} \tag{16}$$

For similar reasons as mentioned above the parameters must satisfy the symmetry restrictions (10). Finally we can differentiate the price function $P(\cdot)$ logarithmically with respect to time to obtain the negative of the rate of technical change:

$$-w_t = \frac{\partial \ln P(\cdot)}{\partial t} = \alpha_t + \beta_{tC} \ln q_C + \beta_{tI} \ln q_I + \beta_{tK} \ln q_K + \beta_{tt} \cdot t\ , \tag{17}$$

where $w_t = \frac{\partial \ln F}{\partial t}$, as measured in (9). It is $-w_t$ the growth of the price function with respect to time, holding prices of consumption, investment and capital constant. For a proof that

$$w_t = \frac{\partial \ln F}{\partial t} = -\frac{\partial \ln P}{\partial t}, \tag{18}$$

see the appendix.

The production function F and the price function P are homogeneous of degree one so that the share equations are homogeneous of degree zero in the quantities or prices, respectively. Conversely, we observe that the shares sum to unity, implying the parameter restrictions, given under (5). From these restrictions, together with the symmetry restrictions (10), we conclude that the share equations are homogeneous of degree zero in the quantities or prices, respectively, and that F and P are homogeneous of degree one. Therefore, a sum of one for the value shares, the restriction (5) and the symmetry restriction (10) provide a test of the hypothesis that the marginal productivity functions (6) and the supply and demand functions (16) can be generated from a production function F or a price function, respectively, that are homogenous of degree one.

To test the properties of well-behaved production and profit functions, we have to develop tests for monotonicity and convexity of these functions. Monotonicity of the production function F requires that

$$\frac{\partial F}{\partial C} \geqq 0 , \frac{\partial F}{\partial I} \geqq 0 , \frac{\partial F}{\partial K} \leqq 0 , \frac{\partial F}{\partial t} \leqq 0 .$$

Monotonicity of the profit function requires that

$$\frac{\partial \pi}{\partial q_C} \geqq 0 , \frac{\partial \pi}{\partial q_I} \geqq 0 , \frac{\partial \pi}{\partial q_K} \leqq 0 , \frac{\partial \pi}{\partial t} \geqq 0 .$$

It follows from (11) that the partial derivatives of the price function have the same sign as those of the profit function. The necessary conditions for monotonicity of the production function are

$$\alpha_C \geqq 0 , \quad \alpha_I \geqq 0 , \quad \alpha_K \leqq 0 , \quad \alpha_t \leqq 0 ; \tag{19}$$

and for the profit function:

$$\alpha_C \geqq 0 , \quad \alpha_I \geqq 0 , \quad \alpha_K \leqq 0 , \quad \alpha_t \geqq 0 . \tag{20}$$

Convexity of the profit function implies convexity of the price function P and vice versa. For tests of convexity see *CJ* (1977). We note here only that after reparametrization the following two inequality restrictions must hold under convexity:

$$\delta_1 \geqq 0 , \quad \delta_2 \geqq 0 . \tag{21}$$

3. Test results for the model of production I

In the preceding section we derived restrictions on the parameters of the system of marginal productivity and supply and demand functions by assuming a translog approximation of the production and profit function, respectively. For hypotheses testing we begin with first-order translog approximations of the marginal productivity and supply and demand functions. We then impose the parameter restrictions implied by integrability of the system to the corresponding translog production or profit function. The stochastic specification of the model is based on the assumption that the true shares have the translog form. In this case we avoid the econometric complications with occur if an error of approximation is included in the error term[3].

The details in developing tests of the theory of production are given in *CJ* (1977). We only summarize the main steps:

1. To test integrability of the marginal productivity and supply and demand functions, we add a stochastic component to the translog functions. As the shares sum to unity, the random variables are not distributed independently and we combine two equations of (6) or (16), respectively, with the equation (8) or (17), respectively, for the rate of technical change.

2. The complete model involves 15 unknown parameters. There are six symmetry restrictions, so that the model involves 9 unknown parameters under the integrability restrictions. Together with the parameter restrictions (5), the symmetry restrictions are also tests of homogeneity of degree one of the underlying production or profit function.

3. Tests of monotonicity and convexity restrictions have been carried out conditional on integrability. We have tested monotonicity and convexity in parallel.

   To test monotonicity, we form t-ratios for the linear hypothesis corresponding to each of these restrictions. We reject monotonicity if the fitted value, say $\hat{\alpha}$, is significantly negative (α_C, α_I) $((\alpha_C, \alpha_I, \alpha_t)$ for the profit function) or positive (α_K, α_t) $(\alpha_K$ for the profit function).

4. To test convexity, a reparametrization of the parameters β_{CC} β_{CI} and β_{II} by the parameters δ_1, δ_2 and λ_{21} is required (see CJ (1977)). In terms of the new parameters necessary conditions for convexity take the form

   $$\delta_1 \geq 0, \delta_2 \geq 0 \ .$$

   We first fit the econometric model with symmetry imposed. To test convexity, we require t-ratios for the linear hypothesis corresponding to each of these inequality restrictions. We reject the hypothesis of convexity if the fitted values, say $\hat{\delta}$, are significantly negative.

5. The estimator of the unknown parameters is based on the method of maximum likelihood. To test the validity of restrictions implied by integrability, we employ test statistics based on the likelihood

[3] See *Simmons* and *Weiserbs* (1979).

ratio λ. Our test statistic is based on $-2 \ln \lambda$ which is under the null hypothesis distributed, asymptotically as chi-squared with a number of degrees of freedom equal to six, the number of symmetry restrictions to be tested. To test the validity of inequality restrictions implied by monotonicity and convexity, we employ test statistics based on the ratio of each inequality constrained coefficient to its standard error. Under the null hypothesis these test statistics are distributed asymptotically as standard normal variables.

6. We set the level of significance for each series at 0.05 and assign a level of significance of 0.025 to the tests of symmetry and of 0.025 to the tests of monotonicity and convexity. The probability of a false rejection for one test among the collection of all tests we consider is less than or equal to 0.05. With the aid of critical values for our test statistics given in Table 1, the reader can evaluate the results of our tests for alternative significance levels. Test statistics for each of the hypotheses implied by the theory are given in Table 2.

Table 1

Critical values of χ^2/degrees of freedom and N (0,1)

Degrees of freedom	Statistic	Level of significance				
		.10	.05	.025	.01	.005
1	N (0,1)	1.28	1.64	1.96	2.33	2.58
5	χ^2/d. f.	1.84	2.21	2.56	3.01	3.35
6	χ^2/d. f.	1.77	2.10	2.41	2.80	3.09
10	χ^2/d. f.	1.6	1.83	2.05	2.32	2.51

We have used the same data as employed by *CJ* (1977, 1978) except for the price and quantity of capital and for the rate of technical change. The latter has been calculated according to (9). The price of capital services q_K has been determined along the lines given in *Jorgenson* and *Hall* (1967) and calculated for the Federal Republic of Germany in *Conrad* and *Jorgenson* (1975). The price of capital depends on the tax rate u, the after tax rate of return r, the investment goods price p_I, the rate of replacement δ, deductable property taxes per unit of capital stock τ and less capital gains from revaluation:

$$q_{K,t} = \frac{1}{1-u_t} \left(p_{I,t-1} r_t + p_{I,t} \delta - (p_{I,t} - p_{I,t-1})\right) + p_{I,t} \tau_t \,.$$

Table 2

Test statistics for translog production and price functions

Hypothesis	Rate with 0-profit[a)]		Average Interest Rate		Bonds Rate	
	Production	Profit	Production	Profit	Production	Profit
Symmetry	1.77	2.37	1.76	1.5	1.96	3.3
Given Symmetry						
Monotonicity						
α_C	200.6	127.9	164.0	130.3	162.3	124.0
α_I	245.2	101.2	138.5	87.4	142.7	100.3
α_K	− 118.5	− 149.3	− 22.1	− 35.0	− 36.5	− 60.7
α_t	− 11.3	11.2	− 13.3	19.8	− 17.5	17.5
Convexity						
δ_1	13.4	19.1	− 19.5	− 5.5	− 2.5	− 1.2
δ_2	0.51	0.03	0.05	0.4	− 0.04	0.2

a) See *C J* (1977).

The price p_I of the acquisition of capital goods differs from the price q_I for the output of investment goods by including indirect taxes[4]. The tax rate has been calculated by dividing taxes on corporate profits and personal income taxes on property income by total property income. Property income does not include imputed wages for self-employed persons and also the income tax has been cleared up by the tax revenue due to labor income[5]. As a proxy for r we tried two alternatives: the rate of return on bonds and a weigthed average of several interest rates on financial assets. The outcome of the test procedure can say something about the sensitivity of the tests with respect to the choice of a rate of return. Profits have been calculated by subtracting $q_{k,t} \cdot K_{t-1}$ from total property income, where K_t is the real private capital stock. If capital costs $q_k K_{-1}$ include profit as a normal return on capital, the corresponding rate of return r_t can be calculated by multiplying the formula for the price of capital $q_{K,t}$ by K_{t-1} and solving for r_t. This is the rate of return employed by *CJ* (1977), which implies $G = 0$ and a degree of homogeneity of $r = 1$.

[4] The prices q_I and p_I are presented in *Conrad* and *Jorgenson* (1975).

[5] See *Conrad* and *Jorgenson* (1975).

The results of our tests of the theory of production based on short-run non-zero profit maximization with labor input given are that the set of restrictions on the parameters of the translog marginal productivity functions implied by integrability cannot be rejected at the corresponding level of significance for both choices of a rate of return. The set of integrability restrictions on the parameters of the translog supply and demand functions cannot be rejected in the case of an average interest rate as a proxy for the rate of return, but has to be rejected in the case of the bonds rate as proxy. Given symmetry, we accept monotonicity of both, the production and profit function, but reject convexity if q_K is based on the average interest rate, as δ_1 is significantly negative at the corresponding level of significance. If the price of capital is based on the bonds rate we also reject convexity of the production function. As we have already rejected symmetry for the profit specification, the test result for convexity is meaningless. Compared with the results under zero profits, the symmetry conditions are less restrictive than the convexity assumption.

4. Model of production II: Separable, homogeneous production function; all inputs as variables

To unify our framework for testing the theory of production under zero and non-zero profits we have assumed that labor input is fixed in the short-run which might be the case with respect to firing but not with respect to hiring labor. The reason for this assumption is, that under zero profits the model specification derived in CJ (1977) coincides with the model specification given in section 2 of this paper.

In this section we assume that all quantities are variables and state the problem as follows:

$$\max_{C,I,K,L} \{\pi = q_C C + q_I I - q_K K - q_L L \mid L^{1/r} = F(C, I, -K, t)\} \tag{22}$$

Necessary conditions for producer equilibrium are:

$$q_C = \lambda F_C, \; q_I = \lambda F_I, \; -q_K = \lambda F_K, \; q_L = \lambda \frac{1}{r} L^{1/r-1}. \tag{23}$$

Again, by employing Euler's theorem, we obtain:

$$\lambda = \frac{\pi + q_L L}{L^{1/r}};$$

and λ substituted into (23) results in the same system of marginal productivity functions as given in (3); for example

$$w_C = \frac{q_C\, C}{\pi + q_L\, L} = \frac{\partial \ln F}{\partial \ln C}, \quad \text{etc.} \tag{24}$$

An additional equation emerges from the marginal productivity condition for labor:

$$\frac{q_L\, L}{\pi + q_L\, L} = \frac{1}{r}. \tag{25}$$

Under positive profits r is greater than unity. This equation can be used to estimate the degree of homogeneity, which enters the equation (9) for the rate of technical change and was already needed to carry out the tests in section 3. We obtained

$r = 1.39$ (0.02)	with q_K based on an average interest rate
$r = 1.27$ (0.01)	with q_K based on the bond rate
$r = 1$	with q_K based on zero profits

We conclude that the marginal productivity function for long-run and short-run profit maximization are the same except for the additional equation (25). Therefore, if we estimate (25) outside the simultaneous system, the outcome of the test will be the same under both models. However, we did not succeed in showing that the same conclusion holds with respect to the dual approach. It is not possible to substitute P or π for F in (24) with three prices instead of three quantities as arguments. We can rewrite (23) as follows:

$$\frac{q_C}{q_L \cdot r} = \frac{F_C(x)}{F(x)}, \quad \frac{q_I}{q_L \cdot r} = \frac{F_I(x)}{F(x)}, \quad -\frac{q_K}{q_L \cdot r} = \frac{F_K(x)}{F(x)},$$

where $(x) = (\frac{C}{L}, \frac{I}{L}, -\frac{K}{L}, t)$. This system can be solved for the three unknown variables $\frac{C}{L}, \frac{I}{L}$ and $\frac{K}{L}$:

$$\begin{aligned} \frac{C}{L} &= f_1\left(\frac{q_c}{q_L \cdot r}, \frac{q_I}{q_L \cdot r}, \frac{q_K}{q_L \cdot r}, t\right) \\ \frac{I}{L} &= f_2(\cdot), \qquad \frac{K}{L} = f_3(\cdot) \end{aligned} \tag{26}$$

L can be obtained from the production function by using (26)

$$L^{\frac{1}{r}} = F(Lf_1, Lf_2, Lf_3, t)$$

or

$$L = \left[G\left(\frac{q_c}{q_L \cdot r}, \frac{q_I}{q_L \cdot r}, \frac{q_K}{q_L \cdot r}, t\right)\right]^{\frac{r}{1-r}}$$

If we insert the optimal production plan into the objective function, then the profit function has the following form:

$$\pi(q_C, q_I, q_K, q_L, t) = q_L \cdot r \left(H\left(\frac{q_C}{q_L \cdot r}, \ldots, t\right) - \frac{1}{r} \right) G\left(\frac{q_C}{q_L \cdot r}, \ldots, t\right)^{\frac{r}{1-r}}$$

or

$$\pi(\cdot) = P(q_C, q_I, q_K, q_L \cdot r, t)$$

with $\pi(\cdot)$ or $P(\cdot)$ homogeneous of degree one in all prices.

This means that we cannot obtain a simple condition on the profit function when the transformation function is separable in $\{L\}$ from $\{C, I, -K, t\}$. Therefore, a supply function, for example for C, is:

$$\frac{\partial \pi(\cdot)}{\partial q_C} = C \quad \text{or,} \quad \frac{\partial \ln \pi}{\partial \ln q_C} = \frac{q_c \cdot C}{\pi}$$

A translog representation of the profit function $\pi(\cdot)$ would now include all four prices. However, the goodness of fit of these supply and demand functions should turn out to be rather poor as the shares are pretty unstable.

5. Model of production III: Implicite production function; all inputs as variables

We finally outline the test procedure for variable quantities without the assumption of explicit separability of L from $\{C, I, K, t\}$, as suggested by *Zweifel* (also 1978).

Given the prices, the profit maximizing problem is

$$\max_{C,I,K,L} \quad \{\pi = q_c C + q_I I - q_K K - q_L L \mid H(\ln C, \ln I, \ln K, \ln L, t) = 0\}$$

The first order conditions are

$$(27) \qquad q_c = -\frac{\lambda H_c}{C}, \quad q_I = -\frac{\lambda H_I}{I}, \quad q_K = \frac{\lambda H_K}{K}, \quad q_L = \frac{\lambda H_L}{L}$$

where $H_C = \dfrac{\partial H}{\partial \ln C}$

From these conditions the following shares for estimation under nonzero profits can be derived, given $H(\cdot)$ in translog form:

$$(28) \qquad \frac{q_C C}{q_L L} = -\frac{H_C}{H_L} = -\frac{\alpha_C + \beta_{CC} \ln C + \beta_{CI} \ln I + \beta_{CK} \ln K + \beta_{CL} \ln L + \beta_{Ct} \cdot t}{\alpha_L + \beta_{LC} \ln C + \beta_{LI} \ln I + \beta_{LK} \ln K + \beta_{LL} \ln L + \beta_{Lt} \cdot t}$$

$$\frac{q_I I}{q_L L} = -\frac{H_I}{H_L} = -\frac{\alpha_I + \beta_{IC} \ln C + \ldots}{\alpha_L + \ldots}$$

$$\frac{q_K K}{q_L L} = \frac{H_K}{H_L} = \frac{\alpha_K + \beta_{KC} \ln C + \ldots}{\alpha_L + \ldots}$$

$$\frac{\pi + q_L L}{q_L L} = -\frac{H_C}{H_L} - \frac{H_I}{H_L} - \frac{H_K}{H_L}$$

The last condition results from (27):

$$q_C C + q_I I - q_K K - q_L L = \lambda [- H_C - H_I - H_K - H_L]$$

or

$$\pi = \lambda H_L \left[- \frac{H_C}{H_L} - \frac{H_I}{H_L} - \frac{H_K}{H_L} - 1 \right] \tag{29}$$

which implies the last share in (28) using the marginal productivity condition for labor in (27). This also implies that the shares obey the adding-up property:

$$\frac{q_C C}{q_L L} + \frac{q_I I}{q_L L} - \frac{q_K K}{q_L L} - \frac{\pi + q_L L}{q_L L} = 0 \, . \tag{30}$$

The equation for the rate of technical change is

$$w_t = \frac{H_t}{H_L} = \frac{\alpha_t + \beta_{tC} \ln C + \ldots \beta_{tt} t}{\alpha_L + \ldots + \beta_{Lt} t} \tag{31}$$

where $w_t = \frac{q_C C}{q_L L} \dot{C} + \frac{q_I I}{q_L L} \dot{I} - \frac{q_K K}{q_L L} \dot{K} - \dot{L}$,

for calculating w_t. For a proof, we differentiate $H(\ln C, \ldots, t)$ totally with respect to time:

$$H_C \dot{C} + H_I \dot{I} + H_K \dot{K} + H_L \dot{L} + H_t = 0 \, .$$

Dividing by H_L and using (28) implies the formula.

For testing the theory of production, we first have to estimate the first three equations in (28) without parameter restrictions and in a second step with symmetry restrictions imposed and the same set of parameters in each denominator[6]. Symmetry of $H_{CI} = H_{IC}$, i. e., implies $\beta_{CI} = \beta_{IC}$ for a translog production function. We finally can test for

[6] There are no parameter restrictions like $\alpha_c + \alpha_I + \alpha_K + \alpha_L \left(1 + \frac{\pi}{q_L L}\right) = 0$ as mentioned by *Zweifel* (1978) besides the fact that such a restriction can not be fulfilled if the profit share varies.

homogeneity of the production function where homogeneity of degree r is defined in the following way:

$$F(\lambda C, \lambda I, \lambda K, \lambda^r L, t) = 0.$$

With $r > 1$ this implies decreasing returns to scale with respect to labor. We differentiate $H(\ln \lambda C, \ln \lambda I, \ln \lambda K, \ln \lambda^r L, t) = 0$ with respect to λ and after setting λ equal to unity we obtain

$$H_C + H_I + H_K + rH_L = 0 .$$

Thus, the last equation in (28) has to be replaced by:

$$\frac{\pi + q_L L}{q_L L} = r \tag{32}$$

In order to satisfy the adding-up property (30), the following parameter restrictions have to be imposed:

$$\begin{aligned}
\alpha_C + \alpha_I + \alpha_K + r\,\alpha_L &= 0 \\
\beta_{CC} + \beta_{IC} + \beta_{KC} + r\,\beta_{LC} &= 0 \\
\beta_{CI} + \beta_{II} + \beta_{KI} + r\,\beta_{LI} &= 0 \\
\beta_{CK} + \beta_{IK} + \beta_{KK} + r\,\beta_{LK} &= 0 \\
\beta_{CL} + \beta_{IL} + \beta_{KL} + r\,\beta_{LL} &= 0 \\
\beta_{Ct} + \beta_{It} + \beta_{Kt} + r\,\beta_{Lt} &= 0 .
\end{aligned} \tag{33}$$

It has to be noticed that the symmetry restrictions can only be interpreted as integrability conditions if the true production function is of the translog type. There are three more functional specifications which imply the same system of marginal productivity functions as given in (28) and (31)[7]. If the hypothesis of integrability is true but we reject $\beta_{ij} = \beta_{ji}$, this only means that the true technology can not globally represented by a translog production function. It does not mean that neoclassical production functions can not generate the system (28) and (31). One production function that generates the same system as (28) and (31) and which is not a monotonic transformation of a translog production function is, for example ($z = (\ln C, \ln I, \ln K, \ln L, t)$)[8]:

$$c \ln (a_0 + \Sigma\, a_j z_j) + b_0 + \Sigma\, b_j z_j = 0 \tag{34}$$

$$\text{where} \quad \alpha_i = c\, a_i + a_0\, b_i \,, \quad \beta_{ij} = a_j\, b_i$$

Of course, the matrix of second-order partial derivatives of the production function (34) is symmetric, but $\beta_{ij} \neq \beta_{ji}$ is consistent with this

7 See *Simmons* and *Weiserbs* (1979) with respect to translog utility functions.

8 This is one of the three functions specification given by *Simmons* and *Weiserbs* (1979).

property. This implies that the symmetry restrictions $\beta_{ij} = \beta_{ji}$ are not the only ones to test the integrability of the system of marginal productivity functions (28). A different set of parameter restrictions exists to garantee integrability of the system (28) to the production function (34). If we therefore reject the hypothesis $\beta_{ij} = \beta_{ji}$, we only reject the translog approximation but not the hypothesis of profit maximization. The integrability hypothesis is true if $\beta_{ij} = \beta_{ji}$ (translog production function) or if any of the integrability restrictions are satisfied which belong to production functions of the type given in (34). Therefore there are at least two different approaches for a choice of a more flexible functional form for testing hypotheses: an approximation of a production function and one of the system of marginal productivity functions. Both approaches can result in the same model to be estimated but this model differs with respect to the parameter restrictions which correspond to the properties of the true model of production. A test of a model of production therefore can only be carried out on the basis of the corresponding choice of approximation.

6. Empirical results for the model of production III

We assume that the translog approximation given in (28) and (31) is an appropriate functional specification of the system of marginal productivity functions. In this case, that is without an approximation error, all parameters in the denominator are equal. Given this equality we test the hypothesis that the system of marginal productivity functions (28) and (31) has been generated by a translog production function. We did not test integrability with respect to the production function (34) by imposing the corresponding parameter restrictions. Given a translog production function we finally test homogeneity of the production function.

As the share equations are homogeneous of degree zero in the parameters we divide the nominator and denominator by α_L and estimate parameters like $\beta^*_{CC} = \beta_{CC}/\alpha_L$, that is, we can not identify α_L. Given equalization of the set of parameters in the denominator there are 29 parameters to be estimated. Symmetry restrictions reduce the number of parameters by ten. We finally impose the six homogeneity restrictions (33) and introduce the parameter r which reduce the number of parameters by five. In Table 3 we present the results of our tests based on the validity of the equality assumption. The price of capital q_K is based on the average interest rate as the rate of return.

As our test statistics exceed by far the critical values given in Table 1, we reject the hypothesis, that the system (28) and (31) represents profit-maximizing marginal productivity functions which have been generated by a translog production function. If we assume the validity of a translog production function, we reject the homogeneity of this function.

Table 3

Test statistics for the translog implicit production function

Hypothesis	Degrees of freedom	Test statistic
Symmetry	Given equality 10	18.5
Homogeneity	Given symmetry 5	4.0

7. Conclusion

We have shown that under alternative assumptions on separability not only functional specifications and sets of parameter restrictions differ extremely but also empirical results. Even with the same model the outcome of the test differs due to the degree of freedom to choose an appropriate price of capital. Furthermore, the problems will increase if we drop the assumption of perfect competition and assume, for example, monopolistic profit maximization. This removes none of our problems of approximation and the price of capital but adds another one, the approximation of supply and demand functions.

We conclude that there is no way to test the theory of production as such. What can be done is only to test a specific model of production under the assumptions made in developing the analytical framework. If *CJ* (1977) accept the theory of production then the correct interpretation of their results is that they accept the theory subject to the assumption of zero profits and explicit separability of the production function. Such a model can be used as a sub-model in a macro-econometric growth model or for further hypothesis testing. One might cast doubt on the validity of the results as *Zweifel* did but empirical results are never generally valid. They are only valid under certain assumptions, conditions, time periods, to mention just a few of the standard limitations. Even if we reject hypotheses based on the non-zero-profit model, there is no reason to consider the test results based on the zero-profit model as irrelevant, because it is very likely that there exists a rate of return based on that even the non-zero-profit model would pass all tests. It is the advantage of the translog approach that it at least tries to minimize the assumptions with respect to the specification of behavioral and technical functions by making it possible to avoid the assumptions of homogeneity, separability or constant elasticity of substitution. The approach does not solve the question about the appropriate price of capital. Therefore, in opposite to *Zweifel*, there is no reason to join *CJ* in their acceptance of the profit maximizing hypothesis if their model will be based on an nonseparable production function and non-zero profits, because the assumption of an arbitrary rate of return does not make the test more general than assuming zero profits, that is, a rate of return consistent with zero profits. We therefore conclude that we have to interprete in any case the test results subject to the assumptions as "tests of a model of production" which also is the title choosen by *CJ*. The meaning of such a methodological approach is that hypothesis testing and parameter fitting should be a joint procedure in constructing econometric models instead of fitting parameters without worrying about the hypotheses made in specifying the equations.

Appendix

We differentiate totally the expression

$$\pi(\cdot) + q_L \cdot L = q_I I + q_C C - q_K K \ ,$$

with respect to t and devide both sides by $\pi(\cdot) + q_L L$:

$$\frac{d \ln (\pi(\cdot) + q_L L)}{dt} = \frac{q_I \cdot I}{\pi + q_L L} \dot{q}_I + \frac{q_C C}{\pi + q_L L} \dot{q}_C - \frac{q_K K}{\pi + q_L L} \cdot \dot{q}_K + $$

$$+ \frac{q_I I}{\pi + q_L L} \dot{I} + \frac{q_C C}{\pi + q_L L} \dot{C} - \frac{q_K K}{\pi + q_L L} \dot{K} \ .$$

$$\text{(A)} \qquad = \sum_{X = C,I,K} (w_X \dot{q}_X + w_X \dot{X}) \ .$$

We next differentiate the logarithm of the profit function (10), including wages, totally with respect to time:

$$\frac{d \ln (\pi(\cdot) + q_L L)}{dt} = \frac{1}{r} \dot{L} + \frac{\partial \ln P(\cdot)}{\partial t} + \sum_{X = I,C,K} \frac{\partial \ln P(\cdot)}{\partial \ln q_X} \cdot \dot{q}_X$$

$$\text{(B)} \qquad = \frac{1}{r} \dot{L} + \frac{\partial \ln P(\cdot)}{\partial t} + \sum_X w_X \cdot \dot{q}_X$$

because of (11) - (13). Comparing (A) and (B), we obtain:

$$\frac{1}{r} \dot{L} + \frac{\partial \ln P(\cdot)}{\partial t} = \sum_X w_X \dot{X} \ .$$

which shows, combined with (9), the validity of (17).

Summary

The objective of the paper is to develop tests of hypotheses from production theory under more general models of production. The zero-profit model will be extended to a non-zero-profit model and the assumption of a separable or homogeneous production function will be dropped. We derive alternative econometric models, based on translog representations of production functions in two outputs and of profit functions in the corresponding prices. We present empirical tests for time series data of West-Germany for 1950 - 1973 and show the sensitivity of the results with respect to weaker assumptions on the underlying production function.

Zusammenfassung

Das Ziel dieses Beitrages ist die Überprüfung produktionstheoretischer Hypothesen auf der Basis allgemeinerer Produktionsmodelle. Das Null-Gewinn-Modell wird zu einem Modell mit Residualgewinn erweitert und die Annahme einer separablen oder homogenen Produktionsfunktion wird fallengelassen. Die entsprechenden ökonometrischen Modelle basieren auf einer translog Darstellung einer Produktionsfunktion in zwei Gütern und zwei Faktoren und einer Gewinnfunktion in den entsprechenden Preisen. Zur Überprüfung der Spezifikationen und Hypothesen werden Zeitreihen für die Bundesrepublik von 1950 - 1973 verwendet. Dabei soll insbesondere die Sensitivität der Testergebnisse bei alternativen Annahmen an die zugrunde gelegte Produktionsfunktion gezeigt werden.

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