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On the Valuation and Analysis of Risky Debt: A Practical Approach Using Rating Migrations

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Abstract

This paper is concerned with the valuation and analysis of risky debt instruments with arbitrary interest and principal payments subject to default risk. We use a discrete risk-neutral present value model with expected payments for risk-neutral investors and risk-free spot rates for the valuation. The expected payments include the potentiality of default by weighting promised payments the risk-neutral default probabilities. The required risk-neutral default probabilities are derived from prices of zero bonds, the current term structure and risk-neutral recovery rates. Based on this debt valuation, we calculate various key figures for analyzing risky debt from the point of view of risk-averse investors (e.g., promised and expected yields, yield spreads, Z-spreads, risk premia). These key figures incorporate the default risk of specific risky debt instruments and therefor lead to improved valuation judgments and valuation results compared to other valuation procedures in theory and practice. Our approach is well-suited for practical applications since the parameters required are easily available from observable data.

Keywords: risky debt, risky debt valuation, expected yield, credit risk model

JEL Classification : G12, G21, G31, G32

This article provides a simple rating-based credit risk model for valuing risky debt. We present both a risk-neutral as well as a risk-adjusted approach to determine the fair price of a risky bond using prices of zero bonds historical rating transition matrices as a starting point. The model is useful for pricing non-callable corporate and government debt subject to default risk and can be used for various risk management purposes. The valuation formulas are simple, and the

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input parameters required for the model are easily estimated using observable data. Furthermore, to our knowledge, this is the first rating-based, reduced-form model to provide full valuation formulas for bond types other than zero-coupon bonds. In other words, our valuation framework can be applied to debt instruments with various kinds of interest and repayment modalities. The generality and practicality of our model should make it particularly attractive to practitioners. In practice, risky debt is not valued separately; instead, the current market value, if available, is used to calculate the implied yield or credit spread of a risky debt instrument. This implied key figure is then compared with the average indicator of other listed bonds of the same rating category and maturity to recognize any mismatches in valuation. In conclusion, a direct valuation of risky debt is not carried out. Instead, the observable market value, as well as average yields or spreads, are applied. As this practice implies the same credit risk for all bonds, it can lead to valuation misjudgements. Our model, on the other hand, allows a practical and realistic valuation that is easy to implement and can be used for risky debt instruments subject to credit risk.

There are numerous credit risk models that can be used for the valuation of risky debt. Previous models can be divided into two broad categories.[1] The first class of models assumes that a stochastic process drives the value of the firm. In these structural models, the firm's debt is modeled as a contingent claim issued against the underlying assets of the firm. Default occurs when the firm value falls below a certain barrier. Structural models were first introduced by Merton (1974). Numerous extensions have been developed to include stochastic interest rates, varying default barriers, or different interest and repayment modalities (*Longstaff/Schwartz* 1992; *Briys/Varenne* 1997; *Geske* 1977). Structural models are challenging to implement in practice since they require estimates for the value and volatility of the firm's assets, which are often not observable.

The second class of models evades this problem. Reduced-form models use ratings and corresponding default probabilities as a starting point to determine credit risk. They model the default event by an exogenous process, which generally does not depend explicitly on the firm's underlying assets. In consequence, reduced-form models do not require estimates for the parameters of the firm's underlying assets. This drastically facilitates the models' applicability in practice.

The first reduced-form models used probabilities of default and recovery rates to evaluate credit risk (*Fons* 1994; *Jarrow/Turnbull* 1995). Many of the earlier approaches model the default process by a doubly stochastic Poisson process with an intensity parameter λ (*Jarrow/Turnbull* 1995; *Madan/Unal* 1998; *Lando* 1998). These models are, therefore, often referred to as intensity-based models.

[1] We refer the reader to *Niklis/Doumpos/Zopounidis* (2018) and *Hao/Alam/Carling* (2010) for recent literature overviews on credit risk modeling.

The earlier models focus explicitly on the transition to default. They do not take potential rating migrations to other rating categories into account. *Jarrow/Lando/Turnbull* (1997) were the first to explicitly incorporate more detailed credit rating information into the valuation methodology. In their discrete-time model, hereinafter referred to as the JLT model, credit risk is incorporated via rating transition matrices, which are modeled using a time-homogeneous Markov chain process with default as the absorbing state. The recovery rates are assumed to be constant. Risk-neutral rating transition probabilities and probabilities of default are derived from historical, risk-averse transition matrices, and then used to value risky debt in a risk-neutral setting.

There are numerous extensions that build on the framework of the JLT model. *Das* and *Tufano* (1995) relax the assumption of a constant recovery rate and make the recovery rate in the event of default stochastic. *Kijima/Komoribayashi* (1998) adapt the calculation of the risk-neutral transition matrices to prevent negative transition probabilities. Recent developments include the relaxation of the assumption of time-homogeneity (*Nickell/Perraudin/Varotto* 2000; *Feng/Gourieroux/Jasiak* 2008) as well as the extension of the model to a continuous-time setting (*Fuertes/Kalotychou* 2006; *Frydman/Schuermann* 2008; *Kadam/Lenk* 2008). We base our model on the JLT approach because its generality and practicality make it especially attractive in practical applications. Due to its discrete-time framework and simplifying assumption regarding the recovery rates, it is easy to implement and highly intuitive. Furthermore, it can be used with all types of term structures. However, we address a shortcoming of the model that constrains its direct applicability in practice. The JLT model and its extensions focus on the valuation of risky zero-coupon bonds. To the best of our knowledge, no model exists for the valuation of bonds with more elaborate interest and repayment structures. We provide a rating-based model that can be used for the valuation of risky bonds with different interest and repayment modalities, making it more applicable to real-world pricing situations. As a rating-based approach, our model is well suited for practical applications. The probabilities of default must only be calculated once per rating category and not individually for each specific bond. Furthermore, unlike structural models, the approach presented in this paper does not require any firm-specific information besides the rating, the seniority, and the collateralization of the debt instrument under consideration.

The paper is organized as follows. Section I recaps risk and bond valuation. Section II presents the model for the risk-neutral valuation of risky bonds. Section III presents the risk-adjusted valuation approach. Section IV illustrates the risk and return analysis. Section V contains a numerical example. Section VI concludes.

I. Risk and Bond Valuation

In the simplest approach to debt valuation, it is assumed that the debt instrument under consideration is risk-free. Under the assumption of arbitrage-free and complete markets, the value of a risk-free debt instrument is determined as the present value of its promised payments to the creditor. Each promised payment is discounted using the spot rate with the same maturity as the corresponding payment.

Let D_0^{rf} be the time $t = 0$ price of a risk-free non-terminable bond with a nominal value, *Norm*, periodic interest payments, I_t, and periodic principal repayments, P_t, at $t = 1, \ldots T$ where T is the (residual) maturity of the bond in years. We can write this as

(1) $$D_0^{rf} = \sum_{t=1}^{T} \frac{I_t + P_t}{\left(1 + r_{0,t}\right)^t}$$

where

$$\sum_{t=1}^{T} P_t = Nom.$$

As can be seen from equation (1), the promised payments in the numerator are discounted using the risk-free spot rates. This is appropriate because there is no risk inherent in the payments of the bond. In reality, bonds can rarely be considered entirely risk-free. On the contrary, the riskiness of different debt instruments varies greatly depending on parameters such as issuer, seniority, or term structure. There are numerous types of risks that bonds are subject to. On the one hand, bonds are subject to market risks. These include the interest rate risk as well as the systematic spread risk.

On the other hand, bonds are subject to bond-specific risks, which are also referred to as unsystematic spread risks. These include the credit risk, which is the possibility of financial losses caused by changes in the credit rating or the default of the bond or its issuer, and the liquidity risk, which refers to the risk that investors might not be able to sell the debt instrument quickly and at an efficient price. We focus on credit risk, especially on default risk, in this article.

When debt instruments are subject to default risk, equation (1) needs to be adapted to account for this risk. There are two main approaches to valuing risky bonds that differ depending on whether the risk is taken into account in the payments the investor is expected to receive (i.e., risk-neutral vs. risk-averse expected cash flows) or in the interest rates used to discount the payments (i.e., risk-free spot rate vs. risk-free spot rate plus risk premium). The first approach, the risk-neutral valuation approach, is primarily used when pricing risky debt instruments. It incorporates the risk inherent in risky debt instruments in the numerator by weighting the promised payments of the bond with risk-neutral probabilities and recovery rates. The resulting expected cash flows represent a pseudo expected value or certainty equivalent. In consequence, they can be discounted using the risk-free interest rate.

The second approach is the risk-adjusted valuation approach. This approach is primarily used in practice (e.g., in project and equity/company valuation). In the risk-adjusted approach, the promised payments of the bond are weighted using historical, or risk-averse, probabilities. The resulting risk-averse expected cash flows are then discounted using a risk-adjusted discount rate, which includes a risk premium on top of the risk-free rate. Both approaches lead to the same valuation results. Both models are the content of the remaining sections.

II. Risk-Neutral Valuation of Risky Bonds

First, we introduce our risk-neutral valuation approach. The discrete-time valuation model we use is based on *Jarrow/Lando/Turnbull* (1997) (JLT), who present a theoretical pricing formula for risky zero-coupon bonds. We extend their valuation framework to risky bonds with more elaborate interest and repayment modalities. Additionally, unlike in the JLT model, we allow the bond to be default prior to maturity. The remaining assumptions our model is based on those outlined in *Jarrow/Lando/Turnbull* (1997). We assume that trading is discrete and that both risk-free and risky bonds of all maturities with different payment structures are traded in the market. Risky bonds can be grouped into different rating categories, and all firms in the same rating category have the same probability of default. The recovery rate is taken to be an exogenously given constant. We assume that markets are complete and that no arbitrage opportunities exist. Furthermore, we assume that the bankruptcy process is uncorrelated with the risk-free spot rates. We impose this assumption to facilitate the empirical investigation. However, as *Jarrow/Turnbull* (1995) demonstrate, it can easily be relaxed, if necessary.

1. Valuation

Let D_0 be the value of a risky bond at time $t = 0$ promising to pay annual interest, I_t, and annual principal repayments, P_t, at $t = 1, \dots T$ where T is the residual term in years. If the firm goes bankrupt, the promised payments may not be paid in full. Instead, the firm will only pay the recovery rate, RR. The recovery rate indicates the size of the payments a creditor receives in the event of default in percent of his outstanding claims. As explained in Section I, in the risk-neutral valuation approach, the value of a risky bond is the present value of the expected cash flows of the risk-neutral investor. We can write this as

$$D_0 = \sum_{t=1}^{T} \frac{E_0'(C_t)}{(1+r_{0,t})^t} \tag{2}$$

where $E_0'(C_t)$ is the risk-neutral expected cash flow and $r_{0,t}$ is the risk-free spot rate.[2] Since we no longer assume risk-free payments but instead incorporate the probability of default in each period t, the expected cash flows now reflect the default risk.

Figure 1 illustrates the possible payment structure of a risky bond. The expected cash flows are derived by weighting the debt instrument's promised payments with the corresponding default probability and survival probabilities to account for default risk.

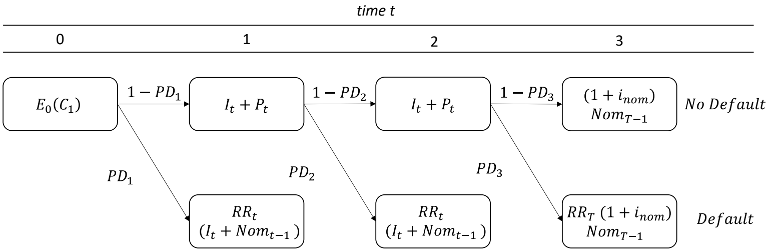

Figure 1. Two state model of a risky debt instrument with remaining term T= 3. The upper strand represents the payments if no default occurs until maturity. The lower strand represents the payments if the debt instrument defaults in any period t. Each state's payments are weighted either with the period-specific probability of default (lower strand) or the probability of survival (upper strand) to determine the expected cash flows in each period.

The expected cash flows incorporate the expected risk-neutral coupon payments as well as the expected risk-neutral redemption payments such that

$$E_0'(C_t) = \prod_{\tau=1}^{t-1}(1-PD_\tau') \cdot [(1-PD_t') \cdot (I_t + P_t) + PD_t' \cdot RR_t'(I_t + Nom_{t-1})] \tag{3}$$

for $t = 1, \ldots, T$. The first term on the right-hand side is the cumulative pseudo survival probability of the debt instrument at the beginning of year t. The term in square brackets is the conditional pseudo-expected value,

[2] Throughout this paper, the hyphen is used to indicate risk-neutrality.

$$E'_{t-1}\left(C_t \,|No\ Default\ until\ t-1\right) = \left(1-PD'_t\right)\cdot\left(I_t + P_t\right) + PD'_t \cdot RR'_t\left(I_t + Nom_{t-1}\right). \tag{4}$$

It represents the expected value of the cash flow at time t under the condition that no default has occurred until $t-1$. Using equations (3) and (4) we can rewrite equation (2) as

$$D_0 = \sum_{t=1}^{T} \frac{\left[\prod_{\tau=1}^{t-1}\left(1-PD'_\tau\right)\right]\cdot E'_{t-1}\left(C_t \mid No\ Default\ until\ t-1\right)}{\left(1+r_{0,t}\right)^t}. \tag{5}$$

It can be seen from (3) that the expected payments of the risk-neutral investor depend on two components. The first component is the risk-neutral, or pseudo, probability of default, PD'_t. This is a conditional probability of default in that it indicates the chance a default will occur at time t given that no default has occurred until $t-1$. The conditional probability of default is calculated from cumulative probabilities of default using Bayes law as

$$PD'_t = Prob'\left(Default\ at\ t \mid No\ Default\ until\ t-1\right) = \frac{CPD'_{0,t} - CPD'_{0,t-1}}{1-CPD'_{0,t-1}} \tag{6}$$

where $CPD'_{0,1} = PD'_1$ and $CPD'_{0,0} = 0$. The cumulative risk-neutral probability of default, $CDP'_{0,t}$, is the probability that default will occur at any time between 0 and t. Conversely, $1-CDP'_{0,t}$, is the cumulative risk-neutral survival probability until time t. The third probability measure we calculate is the total probability of default, which is derived from the cumulative probabilities of default as

$$Prob'\left(No\ Default\ until\ t-1\ and\ Default\ at\ t\right) = CPD'_{0,t} - CPD'_{0,t-1}. \tag{7}$$

As can be seen from (7), the total default probability is the probability that the debt instrument will not default until time $t-1$ and that it will default at time t.

The second component that influences the value of the expected cash flows is the risk-neutral, or pseudo, recovery rate, RR'_t. The recovery rate indicates the size of the payments a creditor receives in the event of default in percent of his outstanding claims. This fraction can depend on the seniority of the risky debt instrument, as well as the value of collateral, for instance. Recovery rates can be determined either historically or implicitly and may vary over time. Historical recovery rates are determined from past defaults and are published regularly by large credit rating agencies (see, for example, *Standard & Poor's Financial Services* (2019) or *Moody's Investors Service* (2011)). Risk-neutral recovery rates are calculated implicitly from market prices (see, for example, *Merton* 1974; *Das/*

Hanouna 2009; *Schläfer/Uhrig-Homburg* 2014).[3] Both components (i.e., probabilities of default and recovery rates) can vary for different rating classes. In rating-based approaches such as the one presented in this paper, it is, therefore, necessary to determine these parameters for each rating category individually.

Since the relationship between cumulative and conditional survival probabilities satisfies

$$1 - CPD'_{0,t} = \prod_{\tau=1}^{t}\left(1 - PD'_{\tau}\right), \tag{8}$$

we can rewrite equation (5) as

$$D_0 = \sum_{t=1}^{T} \frac{\left(1 - CPD'_{0,t-1}\right) \cdot E'_{t-1}\left(C_t \mid No\ Default\ until\ t-1\right)}{\left(1 + r_{0,t}\right)^t}. \tag{9}$$

Under the maintained assumption of complete markets and no arbitrage opportunities, for any specific debt instrument with annual interest payments, I_t, and annual principal repayments, P_t, we can write the value of the risky bond at time $t = 0$ as

$$D_0 = \sum_{t=1}^{T} \frac{\left(1 - CPD'_{0,t}\right) \cdot \left(I_t + P_t\right) + \left(1 - CPD'_{0,t-1}\right) \cdot PD'_t \cdot RR'_t \cdot \left(I_t + Nom_{t-1}\right)}{\left(1 + r_{0,t}\right)^t}, \tag{10}$$

or, using only cumulative probabilities of default, as

$$D_0 = \sum_{t=1}^{T} \frac{\left(1 - CPD'_{0,t}\right) \cdot \left(I_t + P_t\right) + \left(CPD'_{0,t} - CPD'_{0,t-1}\right) \cdot RR'_t \cdot \left(I_t + Nom_{t-1}\right)}{\left(1 + r_{0,t}\right)^t}. \tag{11}$$

The novelty of the valuation formulas in equations (11) and (12) is that they can be used to value debt instruments with different repayment modalities. Our valuation approach can be used to value a wide range of the actual bonds found in the market, since the valuation is not limited to zero-coupons bonds. Furthermore, also the determination of the risk-neutral probabilities of default is straightforward, as we will show in the following subsection. This makes the valuation approach presented in this paper a valuable tool in practical applications.

[3] For a detailed review of the incorporation of recovery rates in credit risk models, we refer the reader to *Altman/Resti/Sironi* (2004).

2. Calculation of Risk-Neutral Probabilities of Default using Zero-Coupon Bonds

In order to price risky coupon bonds using equations (11) or (12), we need to determine the rating-specific risk-neutral probabilities of default. The probabilities of default are derived from the prices of risky zero-coupon bonds with different residual maturities using a bootstrapping procedure. The prices of the zero-coupon bonds can either be observed directly from the market or calculated using the observed yields or yield spreads for different rating classes. The bootstrapping procedure we use in this paper is based on *Jarrow/Lando/Turnbull* (1997). *Jarrow/Lando/Turnbull* (1997) derive the risk-neutral probabilities of default from their formula for the valuation of risky zero-coupon bonds which they write as

$$D_0 = \frac{\left[1-\left(1-RR_T'\right)\cdot CPD_{0,T}'\right]\cdot Nom}{\left(1+r_{0,T}\right)^T}. \tag{12}$$

Rearranging equation (13) for $CPD_{0,T}'$, the risk-neutral cumulative probability of default for a specific rating class according to the JLT model is

$$CPD_{0,T}' = \frac{1}{1-RR_T'}\cdot\left[1-\left(1+r_{0,T}\right)^T\cdot\frac{D_0}{Nom}\right]. \tag{13}$$

The JLT valuation formula is based on the assumption that the zero-coupon bond can only default at maturity. We relax this assumption in order to incorporate more realistic considerations into our model. We assume that the zero-coupon bond can also default at any time before maturity. In such a case, the creditors will receive the recovery rate immediately upon default. We determine the risk-neutral probabilities of default based on equation (11). For risky zero-coupon bonds, it holds that $I_t=0$ and $P_t=0$. In consequence, we can write the value of a risky zero-coupon bond at $t=0$ as

$$\begin{aligned} D_0 = &\sum_{t=1}^{T-1}\frac{\left(1-CPD_{0,t-1}'\right)\cdot PD_t'\cdot RR_t'\cdot Nom}{\left(1+r_{0,t}\right)^t} \\ &+\frac{\left(1-CPD_{0,T-1}'\right)\cdot\left[1-\left(1-RR_T'\right)\cdot PD_T'\right]\cdot Nom}{\left(1+r_{0,T}\right)^T}. \end{aligned} \tag{14}$$

We rearrange equation (15) such that

$$PD_t' = \frac{1}{1-RR_t'}\cdot\left\{1-\frac{\left(1+r_{0,t}\right)^t}{1-CPD_{0,t-1}'}\cdot\left[\frac{D_0}{Nom}-\sum_{\tau=1}^{t-1}\frac{\left(1-CPD_{0,\tau-1}'\right)\cdot PD_\tau'\cdot RR_\tau'}{\left(1+r_{0,\tau}\right)^\tau}\right]\right\}. \tag{15}$$

The term in the curly brackets on the right-hand side of equation (16) corresponds to the expected loss at time t in percent of the total claims. Equation (16) clearly illustrates that the conditional probability of default depends on the risk-neutral recovery rate, RR_t', as well as the expected loss.

Next, we show how to use the procedure to bootstrap the risk-neutral conditional probabilities of default from equation (16). We start in period $t=1$. Given that $CPD_{0,0}' = 0$, we can determine the conditional probability of default for $t=1$ as

$$PD_1' = \frac{1}{1-RR_1'} \cdot \left\{1-\left(1+r_{0,1}\right)\cdot\frac{D_0}{Nom}\right\}.$$

In period $t=2$, given PD_1' and $CPD_{0,1}' = PD_1'$, we get the conditional probability of default for $t=2$ as

$$PD_2' = \frac{1}{1-RR_2'} \cdot \left\{1-\frac{\left(1+r_{0,2}\right)^t}{1-CPD_{0,1}'}\cdot\left[\frac{D_0}{Nom}-\frac{PD_1'\cdot RR_1'}{\left(1+r_{0,1}\right)}\right]\right\}.$$

Next, we calculate the cumulative probability of default for $t=2$, CPD_2, by rearranging equation (6) such that

$$CPD_{0,2}' = PD_2' \cdot \left(1-CPD_{0,1}'\right) + CPD_{0,1}'.$$

We can then determine the conditional probability of default for $t=3$, PD_3, by inserting the parameters calculated hitherto into equation (16).

The process is repeated until $t=T$ to determine the remaining conditional and cumulative risk-neutral default probabilities. The bootstrapping procedure must be carried out separately for each rating category. For example, to determine the risk-neutral probabilities for rating category A, only zero-coupon bonds with an A-rating can be used. These resulting probabilities of default can then be used to determine the fair value of a risky bond with a rating of A at $t=0$.

III. Risk-Adjusted Valuation of Risky Bonds

Next, we introduce our risk-adjusted valuation approach. As mentioned in Section I, the risk-adjusted valuation of risky bonds uses the expected cash flows of the risk-averse investor, $E_0(C_t)$, which are calculated by weighting the promised payments of the debt instrument using the historical probabilities of default and the historical recovery rates. However, when these historical parameters are used to weight the promised cash flows, the resulting risk-averse expected cash flows do not fully factor in the risk inherent in the bond. In consequence, they cannot be discounted using only the risk-free spot rates. Instead, a maturity-dependent risk premium, RP_t, must be added to the risk-free spot rates in order to account for the risk of the debt instrument.

Let D_0 be the value of a risky bond at time $t = 0$. Using the risk-adjusted valuation approach, we can write

$$D_0 = \sum_{t=1}^{T} \frac{E_0(C_t)}{\left(1 + r_{0,t} + RP_t\right)^t} \tag{16}$$

where

$$E_0(C_t) = \prod_{\tau=1}^{t-1} (1 - PD_\tau) \cdot \left[(1 - PD_t)(I_t + P_t) + PD_t \cdot RR_t \cdot (I_t + Nom_{t-1})\right], \tag{17}$$

PD_t is the risk-averse probability of default, and RR_t is the risk-averse recovery rate for period t. Unlike the risk-neutral probabilities of default used in Section II, which we must bootstrap from risky zero-coupon bonds, the risk-averse probabilities of default are derived from historical market data. Risk-averse probabilities of default are published regularly by rating agencies in so-called rating transition matrices or rating migrations (see, for example, *Standard & Poor's Financial Services* (2019) or *Moody's Investors Service* (2011)). A t-year rating transition matrix is a table listing the cumulative probabilities that an issuer stays within a specific rating category, transitions to another rating category, or defaults until the end of the t-year period. The rating category the debt instrument is in at the beginning of the period under consideration is indicated in the headers of the rows, while the rating category the debt instrument is in at the end of the period is indicated in the column headers.

The parameters in the numerator of equation (17) are readily available so that investors can determine the risk-averse expected cash flows. However, investors are also interested in the maturity-dependent risk premia, RP_t. These risk premia reflect the systematic risk of the debt instrument and are required to calculate the present value of the risk-averse expected cash flows. They can vary over time and are expressed in % p.a. There are several methods that can be used to determine risk premia. For example, we can calculate the risk premia using the Capital Asset Pricing Model (CAPM) (*Treynor* 1962; *Sharpe* 1964; *Lintner* 1965a, b; *Mossin* 1966). Alternatively, it is possible to derive the risk premia using the prices of risky bonds obtained from risk-neutral valuation models such as the JLT model or the approach presented in Section II of this paper.

1. Risk Premia for Risky Zero-Coupon Bonds

For zero-coupon bonds, the approach is straightforward. Recall from Section I that, under the assumption of arbitrage-free and complete markets, both the risk-neutral and the risk-averse approach must lead to the same fair value for a risky bond. We can use this insight and equation (15) to determine the prices of the risky bonds, D_0, on the left side of equation (17). Additionally, we know the risk-free spot rates and can calculate the risk-averse expected cash flows using the historical default probabilities and recovery rates. The only missing parameters in equation (17) are the risk premium for each period t, $RP_{t.}$

First, we show how to use the JLT model to determine the risk premia. For this, we adapt the risk-neutral valuation approach presented in *Jarrow/Lando/Turnbull* (1997) to the risk-adjusted setting. We write the adapted formula for the price of a risky zero-coupon bond as

$$D_0 = \frac{\left[1-\left(1-RR_t\right)\cdot CPD_{0,T}\right]\cdot Nom}{\left(1+r_{0,T}+RP_T\right)^T} \tag{18}$$

and then rearrange the equation such that

$$RP_T = \sqrt[T]{\frac{Nom}{D_0}\cdot\left[1-\left(1-RR_T\right)\cdot CPD_{0,T}\right]}-\left(1+r_{0,T}\right) \tag{19}$$

to calculate the risk-premium of the zero-coupon bond.

As mentioned in Section II, a significant drawback of the JLT approach is that the valuation formula is based on the assumption that the zero-coupon bond can only default at maturity. In our model, we assume that the zero-coupon bond can default prior to maturity. The price of a risky zero-coupon bond based on our risk-adjusted valuation approach is given by

$$\begin{aligned} D_0 &= \sum_{t=1}^{T-1}\frac{\left(1-CPD_{0,t-1}\right)\cdot PD_t\cdot RR_t\cdot Nom}{\left(1+r_{0,t}+RP_t\right)^t} \\ &+\frac{\left(1-CPD_{0,T-1}\right)\cdot\left[1-\left(1-RR_t\right)\cdot PD_T\right]\cdot Nom}{\left(1+r_{0,T}+RP_T\right)^T}. \end{aligned} \tag{20}$$

Equation (22) differs from its risk-neutral counterpart (15) in that expected payments in the numerator are calculated by weighting the promised payments with the risk-averse probabilities of default and recovery rates, PD_t and RR_t, rather than the risk-neutral parameters. Conversely, the denominator in the risk-adjusted valuation in (22) incorporates a risk premium, RP_t, on top of the spot rate, $r_{0,t}$, while the risk-neutral approach does not. Furthermore, we now not only have one single risk premium for the zero-coupon bond with maturity T but different varying risk premia for each time $t = 1, 2, \ldots, T$ to reflect the default risk correctly. In consequence, we must use a bootstrapping technique to calculate these time-dependent risk premia. For this, we rearrange equation (22) such that

$$(21) \qquad RP_t = \sqrt[t]{\frac{\left(1 - CPD_{0,t-1}\right) \cdot \left[1 - \left(1 - RR_t\right) \cdot PD_t\right]}{\frac{D_0}{Nom} - \sum_{\tau=1}^{t-1} \frac{\left(1 - CPD_{0,\tau-1}\right) \cdot PD_\tau \cdot RR_t}{\left(1 + r_{0,\tau} + RP_\tau\right)^\tau}}} - \left(1 + r_{0,t}\right).$$

For the bootstrapping technique, we start in period $t = 1$. Given that $CPD'_{0,0} = 0$, we can determine the risk premium for $t = 1$ as

$$RP_1 = \frac{\left[1 - \left(1 - RR_1\right) \cdot PD_1\right]}{\frac{D_0}{Nom}} - \left(1 + r_{0,1}\right).$$

In period $t = 2$, for the risk premium we can write

$$RP_2 = \sqrt[2]{\frac{\left(1 - CPD_{0,1}\right) \cdot \left[1 - \left(1 - RR_2\right) \cdot PD_2\right]}{\frac{D_0}{Nom} - \frac{PD_1 \cdot RR_1}{1 + r_{0,1} + RP_1}}} - \left(1 + r_{0,2}\right).$$

This process is repeated until $t = T$. Again, the risk premia must be calculated separately for each rating category.

2. *Risk Premia for Risky Coupon Bonds*

When determining the risk premia of risky debt instruments other than zero-coupon bonds, investors must first determine the expected future prices of the corresponding risky debt instrument. Let $E_0\left(D_{t^+} \mid No\ Default\ until\ t\right)$ be today's (i.e., at time $t = 0$) expected price of a debt instrument at time t^+ (i.e., directly after the interest and principal payments are paid at time t). As with the current fair price for a debt instrument, we get the same expected future fair price irrespective of whether we use the risk-neutral or the risk-averse valuation approach. In the risk-neutral valuation approach, the expected price of a bond is given by

$$E_0\left(D_{t+}\,|No\ Default\ until\ t\right)$$

$$(22)\qquad = \frac{\left(1-PD'_{t+1}\right)\cdot\left[I_t + P_t + E'_0\left(D_{(t+1)^+}\,|No\ Default\ until\ t+1\right)\right]}{1+E_0\left(r_{t,t+1}\right)} + \frac{PD'_{t+1}\cdot RR'_t\cdot\left(I_t + Nom_{t-1}\right)}{1+E_0\left(r_{t,t+1}\right)}$$

where

$$E'_0\left(D_{T+}\,|\,No\ Default\ until\ T\right)=0$$

and $E_0\left(r_{t,t+1}\right)$ is the expected future spot rate from t to $t+1$. Any desired risk-free term structure model can be used to determine the expected future spot rates. Numerous estimation techniques have been developed to determine future risk-free spot interest rates.[4] In this paper, we derive the expected future spot rates based on the assumption that the Pure Expectations Hypothesis holds. The Pure Expectations Hypothesis postulates that the current term structure fully incorporates all information on the future development of the interest rates. Using the Spot-Forward-Relation, the future rates can be derived as the geometric mean of the current interest rates. We can write this as

$$(23)\qquad E_0\left(r_{t,t+1}\right)=\frac{\left(1+r_{0,t+1}\right)^{t+1}}{\left(1+r_{0,t}\right)^{t}}-1.$$

We determine the expected future prices backward, starting in period $t=T-1$. Once we have determined all expected future bond prices, we can calculate the risk-premia of risky coupon bonds. For this, we write the value at of a risky bond at time $t=0$ in the risk-adjusted valuation setting as

$$D_0 = \sum_{\tau=1}^{t-1}\frac{\left(1-CPD_{0,\tau-1}\right)\cdot\left[\left(1-PD_\tau\right)\cdot\left(I_\tau + P_\tau\right)+PD_\tau\cdot RR_\tau\cdot\left(I_\tau + Nom_{\tau-1}\right)\right]}{\left(1+r_{0,\tau}+RP_\tau\right)^{\tau}}$$

$$(24)\qquad +\left(1-CPD_{0,t-1}\right)\cdot\frac{\left(1-PD_t\right)\cdot\left[I_t + P_t + E_0\left(D_{t+}\,|\,No\ Default\ until\ t\right)\right]+PD_t\cdot RR_t\cdot\left(I_t + Nom_{t-1}\right)}{\left(1+r_{0,t}+RP_t\right)^{t}}$$

4 We refer the reader to *Marangio/Massim/Ramponi* (2002) for an overview of the spot rate estimation literature.

where

$$E_0\left(D_{T+} \mid No\ Default\ until\ T\right) = 0.$$

We then rearrange (26) such that

(25)

$$RP_t = \sqrt[t]{\frac{(1-CPD_{0,t-1})\cdot\left\{(1-PD_t)\cdot\left[I_t + P_t + E_0\left(D_{t+} \mid No\ Default\ until\ t\right)\right] + PD_t \cdot RR_t \cdot (I_t + Nom_{t-1})\right\}}{D_0 - \sum_{\tau=1}^{t-1}\frac{(1-CPD_{0,\tau-1})\left[(1-PD_\tau)\cdot(I_\tau + P_\tau) + PD_\tau \cdot RR_\tau \cdot (I_\tau + Nom_{\tau-1})\right]}{(1+r_{0,\tau}+RP_\tau)^\tau}}} - (1+r_{0,t}).$$

The risk premia can be derived using a bootstrapping method starting in period $t=1$ up to period $t=T$.

IV. Risk and Return Analysis

An important application of the model presented in this paper is in the area of risk management, where it can be used to determine various risk and return parameters. For example, it can be used to compute the following three commonly used statistics in practice: yield to maturity, yield spread, and Z-spread. In practice these key indicators are derived using the current market price of a risky debt instrument. When determining the yield of a risky bond, investors focus predominantly on the promised yield to maturity, y_T, which is the yield of the risky debt instrument in the case that no default occurs. It is calculated based on the promised payments of the bond by solving

$$D_0 = \sum_{t=1}^{T}\frac{I_t + P_t}{(1+y_T)^t} \tag{26}$$

for y_T. This maturity dependent implied yield is used in practice to compare different risky debt instruments within the same rating category and maturity. This approach's advantage is that no insight into risk neutral default probabilities and recovery rates is required when calculating the indicator. However, this method should only be used if the bond to be analyzed features the same probability of default and recovery rate as reflected in the peer group's average key figures (yields or spread) of the same rating category. If this is not the case, the method may in practice lead to valuation misjudgements or mispricing. Since it does not take a potential default of the debt instrument into account, the promised yield to maturity is the upper bound on a debt instrument's actual yield. In consequence, it may be significantly too high to be used in risk management considerations. It may, therefore, be advisable to additionally determine the expected, or implied yield to maturity, $E_0(y_T)$. The expected yield is derived based on the expected risk-averse cash flows of the risky bond by solving

(27) $$D_0 = \sum_{t=1}^{T} \frac{E_0(C_t)}{(1 + E_0(y_T))^t}$$

for $E_0(y_T)$ with

$$E_0(C_t) = (1 - CPD_{0,t-1}) \cdot [(1 - PD_t) \cdot (I_t + P_t) + PD_t \cdot RR_t \cdot (I_t + Nom_{t-1})].$$

The expected yield takes the risk-averse probabilities of default and recovery rates into account and is, therefore, a more realistic estimate for the actual yield of a risky bond. The expected yield reflects now a more accurate picture of a bond's default risk as it takes into account bond specific probabilities of default as well as recovery rate.

Based on the yields, it is possible to determine the yield spread. The yield spread measures the difference between the yield of a risky debt instrument and the yield of an otherwise identical risk-free debt instrument. Investors can again differentiate between the promised and expected yield spread. The promised yield spread, YS_T, is the difference between the promised yield to maturity and the yield of a risk-free bond of identical maturity.

(28) $$YS_T = y_T - y_T^{rf}$$

The expected yield spread, $E_0(YS_T)$, on the other hand, is derived using the expected yield to maturity as

(29) $$E_0(YS_T) = E_0(y_T) - y_T^{rf}.$$

Finally, it may also be of interest to investors to determine the zero-volatility spread, or Z-spread, of a risky debt instrument. The Z-spread is a constant credit spread that is added to each point on the term structure when discounting the cash flows of the bond, which ensures that the cash flows of the debt instrument equal its current fair price. In other words, it corresponds to the parallel shift of the term structure that is required to make the present value of the debt instrument's cash flows equal to its market price. The promised Z-spread, ZS_T, is calculated based on the promised payments of the bond. It is derived by solving

(30) $$D_0 = \sum_{t=1}^{T} \frac{I_t + P_t}{(1 + r_{0,t} + ZS_T)^t}$$

for ZS_T. The expected Z-spread, $E_0(ZS_T)$, is calculated based on the risk-averse expected payments from

(31) $$D_0 = \sum_{t=1}^{T} \frac{E_0\left(C_t\right)}{\left(1 + r_{0,t} + E_0\left(ZS_T\right)\right)^t}.$$

When determining the yield, the yield spread, or the Z-spread, it is important to determine the fair price of the risky debt on the left-hand side of all equations (28 – 33) as accurately as possible.

V. Numerical Example

This section lays out our formulas for the rating-based valuation of risky debt using a step-by-step example. First, we illustrate the calculation of risk-neutral probabilities of default. Using these risk-neutral default rates, we then show how to value risky bonds using this article's risk-neutral and risk-averse valuation frameworks. The example is based on two fictitious coupon bonds. Each bond has a nominal value of 100, a fixed coupon of 4 % p. a., and a residual term of three years. The two bonds are identical except for their ratings. While the first bond belongs to rating category A at time $t = 0$, the second bond has an initial rating of B. In total, there are three credit rating categories. The highest rating category is denoted by "A", the second-highest rating category is "B", and the third rating category is "D", which stands for default. The parameters of the risky bonds are summarized in Table 1.

Table 1

Parameters of the Risky Bonds

Parameter	*Symbol*	*Value*
Term	T	3
Nominal value	Nom_0	100.00
Nominal interest rate	i_{nom}	4.00 %
Rating	A or B	

This table reports the parameters of the two risky coupon bonds, which are used in the numerical examples. The bonds are identical except for their initial ratings. While the first bond has an initial rating of A, the second bond is initially rated in category B of our three-part rating scale (A, B, and D).

In addition to these parameters, we require information on the rating migrations the two bonds can potentially undergo. Such information is summarized in rating transition matrices, which are published regularly by credit rating agencies (see, for example, *Standard & Poor's Financial Services* (2019) or

Moody's Investors Service (2011)). The estimates in such rating transition matrices are obtained from historical observations of credit rating changes. In other words, the credit ratings of a fixed group of firms are observed at the beginning and at the end of a particular time period and then summarized in the rating transition matrix. Since only the very beginning and the end of the period are compared, rating transition matrices do not include the exact timing of any transitions within the period. This means that rating migrations are based on the assumption that every firm has made either exactly one transition or has not transitioned at all throughout the specified period.

In our example, we use a fictitious one-year rating transition matrix, which contains the cumulative probabilities from $t = 0$ until $t = 1$. These one-year matrices can be obtained from rating agencies. To facilitate our calculations, we assume that the one-year transition probabilities follow a time-homogeneous Markov process. This means that the probabilities for a transition to another state are the same for each one-year period in $t = [0, T]$. The matrix contains fictitious risk-averse transition and default probabilities for bonds with ratings A, B, or D.

Figure 2 illustrates the necessity of incorporating rating migrations when calculating cumulative probabilities of default for longer periods.

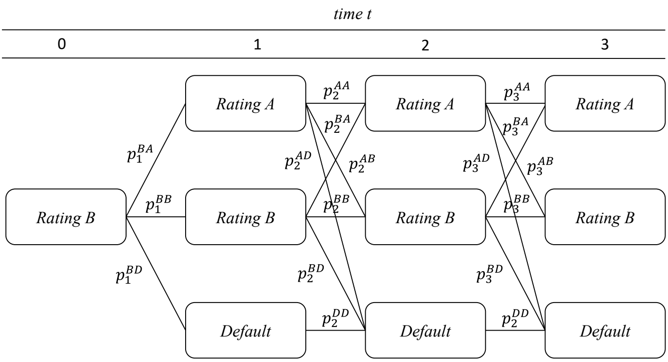

Figure 2. Rating migration model of a debt instrument with an initial rating B and a residual term $T = 3$ and possible ratings A, B and D, where D represents default. The probability $p^{Rating\,t-1,\,Rating\,t}$ represents the probability of a rating migration in period t.

The one-year rating transition matrix is shown in Table 2. Examining the first row, the probability of staying in the highest credit rating category A over a period of one year is 0.9. The transition rate from the highest rating category to the second-highest rating category, B, is 0.06, and the probability that a firm in the highest rating category will default before the end of the one-year period is

0.04. This rate of default does not take into account the possibility that the firm is first downgraded (occurring with a probability of 0.06) and then subsequently defaults (occurring with a rate of 0.1 in rating category B) within the one-year period. As can be seen from the last row in the rating transition matrix, for simplicity of estimation, we assume that bankruptcy (state D) is an absorbing state. This means that once a bond defaults, it can no longer be upgraded again to the higher rating categories A and B.

Table 2

One-Year Rating Transition Matrix

1-year	*A*	*B*	*D*
A	0.9000	0.0600	0.0400
B	0.1000	0.8000	0.1000
D	0.0000	0.0000	1.0000

This table reports the one-year rating transition matrix used in the numerical example. The transition matrix is based on a three-part rating scale with the ratings A, B, and D. A corresponds to the highest rating, B corresponds to the second-highest rating, and D corresponds to default.

Since the two bonds used in our examples both have a maturity of three years, we are also interested in the two- and three-year rating transition matrices. Table 3 illustrates the multi-period transition matrices, which contain the cumulative probabilities from $t=0$ until $t=2$ and $t=3$, respectively. Since the one-year transition matrix is time-homogenous, we calculate the two-year transition matrix by multiplying the one-year matrix by itself. The three-year rating matrix is derived by multiplying the resulting two-year matrix with the initial one-year rating transition matrix. All transition probabilities follow a first-order Markov process. Therefore, each transition probability only depends on the previous period.

Table 3

Two-Year and Three-Year Rating Transition Matrices

two-year	*A*	*B*	*D*
A	0.8160	0.1020	0.0820
B	0.1700	0.6460	0.1840
D	0.0000	0.0000	1.0000

three-year	*A*	*B*	*D*
A	0.7446	0.1306	0.1248
B	0.2176	0.5270	0.2554
D	0.0000	0.0000	1.0000

This table reports the two- and three-year rating transition matrices derived from the one-year rating transition matrix in Table 2. The two- and three-year transition matrices are derived by multiplying the one-year rating transition matrix one and two times by itself, respectively. The transition matrix is based on a three-part rating scale with the ratings A, B, and D. A corresponds to the highest rating, B corresponds to the second-highest rating, and D corresponds to default.

To be able to value the bonds specified in Table 1, we need additional input parameters. These input parameters are summarized in Table 4. First, we need the risk-free spot rates,$r_{0,t}$, for $t = 1, 2, 3$. In this numerical example, we use a fictitious normal term structure. Second, we require the prices of risky zero-coupon bonds for the rating categories A and B in order to calculate the risk-neutral probabilities of default for the two rating classes. In our examples, we determine the risky zero-coupon prices using the yield of risky zero-coupon bonds with residual maturities of 1, 2, and 3 years based on (28). Table 4 contains exemplary yields for ratings A and B. In practice, the market prices of the zero-coupon bonds can be used directly, if available. Otherwise, the yields of zero-coupon bonds with different ratings observable in the market should be used.

Third, we need the risk-neutral and risk-averse recovery rates. For both rating categories A and B, we assume constant risk-neutral and the risk-averse recovery rates of 0.55 and 0.75, respectively. This assumption is imposed to simplify the estimation. It is equivalent to the assumption that both bonds have the same seniority. In practice, the recovery rates of bonds with the same seniority but different ratings may differ.

Table 4

Input Parameters for the Valuation

Time t	*1*	*2*	*3*
Risk-free Spot Rates $r_{0,t}$	1.00 %	1.50 %	2.00 %
Yield of Risky Zero-Coupon Bond (Rating A) γ_A	2.50 %	3.50 %	5.00 %
Yield of Risky Zero-Coupon Bond (Rating B) γ_B	4.00 %	5.00 %	6.50 %
Risk-Neutral Recovery Rate RR_t'	0.55	0.55	0.55
Risk-Averse Recovery Rate RR_t	0.75	0.75	0.75

This table reports the additional input parameters used for the valuation of the risky coupon bonds described in Table 1. The recovery rates are independent of the bonds' rating.

1. *Risk-Neutral Probabilities of Default*

This section demonstrates how we calculate the risk-neutral probabilities of default for rating categories A and B using the yields of the risky zero-coupon bonds specified in Table 3. We calculate the risk-neutral probabilities of default based both on our valuation formulas (FKW) as well as using the approach presented in *Jarrow/Lando/Turnbull* (1997) (JLT). The results are shown in Table 5.

Table 5

Risk-Neutral Probabilities of Default

	Model	*FKW*			*JLT*		
	Time	*1*	*2*	*3*	*1*	*2*	*3*
Rating A	Cumulative PD_t'	3.25 %	8.58 %	18.91 %	3.25 %	8.51 %	18.51 %
	Total PD_t'	3.25 %	5.33 %	10.32 %	3.25 %	5.25 %	10.00 %
	Conditional PD_t'	3.25 %	5.51 %	11.29 %	3.25 %	5.43 %	10.93 %
Rating B	Cumulative PD_t'	6.41 %	14.72 %	27.70 %	6.41 %	14.57 %	27.00 %
	Total PD_t'	6.41 %	8.31 %	12.97 %	6.41 %	8.16 %	12.43 %
	Conditional PD_t'	6.41 %	8.88 %	15.21 %	6.41 %	8.72 %	14.55 %

This table reports the risk-neutral probabilities of default derived using the approach presented in this paper (FKW) as well as based on the valuation approach proposed by *Jarrow; Lando; and Turnbull* (1997) (JLT). For each model, the table shows the cumulative, total, and conditional risk-neutral probabilities of default derived for each period for bonds with rating A as well as for bonds with rating B.

Recall that in this paper, we make the assumption that the risky debt instrument can default at any point before maturity. *Jarrow/Lando/Turnbull* (1997), on the other hand, assume that default can only occur at maturity. As can be seen from Table 5, these different assumptions have an impact on the probabilities of default derived using the two models. For rating category A, all probabilities of default calculated using our approach are strictly greater than the default probabilities derived using the JLT model.

2. Bond Valuation

In this section, we value the two risky bonds utilizing the risk-neutral and risk-averse procedures described in this paper. We value both bonds three times, assuming a different repayment agreement (lump-sum, constant principal, annuity) in each round. For comparison purposes, we also determine the price of the two bonds assuming they are risk-free based using equation (1) for all three repayment cases. Tables 6 and 7 only contain one risk-free value for both bonds. This is the case because the bonds are identical when we disregard the ratings of the two bonds and assume that both are risk-free.

In our first valuation round, we assume that both bonds feature lump-sum repayment. Table 6 summarizes the valuation results. The value of the risk-free bond is 105.88. To calculate the value of the risky bonds, we first determine the expected cash flows. We calculate the risk-neutral and risk-averse expected cash flows using equations (3) and (18), respectively. Next, we determine the value of the two risky bonds using our risk-neutral valuation framework in equation (11). The value of the risky bond with rating A is 97.22, while the price of the B-rated bond is 93.11. The price of the A-rated bond is higher because the probabilities of default are lower for this rating category. Finally, we are also interested in the risk-premia required in the risk-averse valuation approach. For this, we first calculate the expected bond prices based on equation (24) using the expected future spot rates derived from equation (25). We then calculate the risk-premia using our bootstrapping technique based on equation (28).

Table 6

Valuation of Risky Lump-Sum Bonds (Ratings A & B)

	Time *t*	*0*	*1*	*2*	*3*
Risk-free	Value of risk-free debt D_0^{rf}	105.88			
Rating A	Value of risky debt D_0	97.22			
	Risk Premium RP_t		0.53 %	0.99 %	2.06 %
	$E_0'(C_t)$		5.73	6.71	90.24
	$E_0(C_t)$		6.96	6.95	94.36
	$E_0(D_{t+} \mid no\ Default\ at\ t)$		95.57	95.83	0.00
Rating B	Value of risky debt D_0	93.11			
	Risk Premium RP_t		0.72 %	1.30 %	2.56 %
	$E_0'(C_t)$		7.41	8.17	82.61
	$E_0(C_t)$		11.40	9.82	83.01
	$E_0(D_{t+} \mid no\ Default\ at\ t)$		92.57	94.05	0.00

This table reports the valuation results for the two bonds described in Table 1. Both bonds are assumed to feature lump-sum repayment. $E_0'(C_t)$ and $E_0(C_t)$ are the risk-neutral and risk-averse expected cash flows, respectively. $E_0(D_{t+} \mid no\ Default\ at\ t)$ is today's (i. e., at time $t = 0$) expected price of a debt instrument at time t^+ (i. e., directly after the interest and principal payments are paid at time t).

As can be seen from Table 6, the expected cash flows at $t = 1$ and $t = 2$ are higher for rating category B than for rating category A. This is due to the fact that the probabilities of default for category B are higher than for category B. In consequence, for rating B, more weight is put on the case that the bond will default and that the recovery rate will be paid to the creditor compared to rating A, making the expected cash flows higher. The risk premia for both rating A and rating B are positive. As mentioned in Section III, in the risk-adjusted approach, the expected cash flows do not fully factor in the risk inherent in the bond. These risk premia must be added to the risk-free spot rates when discounting the risk-averse cash flows in order to account for the risk of the debt instrument.

In our second and third valuation round, we repeat the valuation but change the underlying assumption on the repayment modality of the two loans to constant principal and annuity repayment, respectively. The value of the risk-free and risky debt is again determined using equations (1) and (11), respectively. We then perform a simple risk and return analysis for the three scenarios. The results are contained in Table 7.

Table 7

Valuation of Risky Bonds with Different Repayment Agreements (Rating A and B)

	Repayment Form	*Lump Sum*	*Constant Principal*	*Annuity*
Risk-free	Value of risk-free debt D_0^{rf}	105.88	104.57	104.61
Rating A	Value of risky debt D_0	97.22	99.87	99.81
	Promised Yield to Maturity y_T	5.02 %	4.07 %	4.10 %
	Expected Yield to Maturity $E_0(y_T)$	3.92 %	2.97 %	3.00 %
	Promised Yield Spread YS_T	3.06 %	2.42 %	2.44 %
	Expected Yield Spread $E_0(YS_T)$	1.95 %	1.32 %	1.34 %
	Promised Z-Spread ZS_T	3.05 %	2.40 %	2.45 %
	Expected Z-Spread $E_0(ZS_T)$	1.96 %	1.35 %	1.37 %
Rating B	Value of risky debt D_0	93.11	97.05	96.96
	Promised Yield to Maturity y_T	6.61 %	5.61 %	5.64 %
	Expected Yield to Maturity $E_0(y_T)$	4.30 %	3.16 %	3.19 %
	Promised Yield Spread YS_T	4.64 %	3.93 %	3.98 %
	Expected Yield Spread $E_0(YS_T)$	2.34 %	1.51 %	1.53 %
	Promised Z-Spread ZS_T	4.63 %	3.96 %	3.99 %
	Expected Z-Spread $E_0(ZS_T)$	2.39 %	1.56 %	1.58 %

This table reports the valuation results and various risk and return parameters for the two bonds described in Table 1 under three different scenarios. In the first valuation round, both bonds are assumed to feature lump-sum repayment. In the second round, both bonds feature constant principal repayment, and in the third round, the bonds have annuity repayment.

The value of both the risk-free and the risky debt varies depending on the repayment modality of the bond. While the value of risk-free debt is highest for lump-sum repayment (105.88), this repayment modality leads to the lowest debt value for both the bond with rating A (97.22) and the bond with rating B (93.11) when the risk is taken into account. The yields to maturity of the two bonds, on the other hand, are highest for lump-sum repayment. Since the promised yields for rating B are generally higher than the respective yields for rating A, also the yield and Z-spreads are higher for rating B than for rating A. An important insight from our model for risk management purposes can be derived from the comparison of the promised yield and Z-spreads with their expected counterparts. When we take the risk of the two bonds into account, the yield and Z-spreads are drastically reduced. The expected spreads of the B-rated bond are approximately 48 – 62 % lower than the promised spreads. For the A-rated bond, the yields are lower by approximately 37 – 44 %. This highlights the fact that investors may significantly overestimate the performance of a risky bond when focusing solely on the promised payments. The promised yields, yield spreads, and Z-spreads are merely the upper limit for their risky counterparts.

VI. Conclusion

This article presents a model for the valuation of risky debt, which uses historical rating transition matrices. The model is based on the seminal paper by *Jarrow/Lando/Turnbull* (1997), and includes both a risk-neutral as well as a risk-adjusted approach to determine the fair price of a risky bond. The model is well-suited for practical applications since it provides simple valuation formulas that can be applied to debt instruments with various kinds of interest and repayment modalities. Furthermore, the input parameters required for the model are easily estimated using observable data. An illustrative example is provided to highlight the easiness of application.

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