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Uncertainty in the Black-Litterman Model: A Practical Perspective

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Abstract

Deriving an optimal asset allocation hinges crucially on the quality of inputs used in the optimization. If the vector of expected returns and the covariance matrix are known with certainty, mean-variance optimization produces optimal portfolios. If, however, these parameters are estimated with uncertainty, mean-variance optimization maximizes estimation error. We provide a literature review of procedures developed in academia to incorporate parameter uncertainty in the asset allocation process, focusing on common heuristics and Bayesian methods. The Black-Litterman model, an application of the Bayesian framework, has practical appeal for investors as it permits the specification of subjective views. Calibration of the model is, however, not trivial and induces rigidity. In *Fuhrer* and *Hock* (2023), a generalization of the Black-Litterman model was introduced and a fully quantitative, objective parameterization was derived. Here, we start with the same generalization, but present a qualitative, more intuitive approach for setting parameters. This gives the investor more control over the mixing of views and equilibrium returns, while lending intuition to the parameter choice in the classical setting.

Keywords: Asset Allocation, Bayesian, Black-Litterman Model, Model Uncertainty, Investment Decisions, Portfolio Choice

JEL Classification: C11, D84, G11

I. Introduction

The classical mean-variance optimization of *Markowitz* (1952) is often considered the standard mathematical framework to derive an optimal asset allocation. From a practical standpoint, however, several problems of unconstrained mean-variance optimization are well documented: Resulting portfolios are often highly-concentrated, very sensitive to the input parameters and maximize esti-

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mation error in the inputs (*Idzorek*, 2007). The problems predominately stem from the assumption that the vector of expected mean returns $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ used for mean-variance optimization are stable and known with certainty. In fact, these input parameters are unknown and can only be estimated with uncertainty, which has to be taken into account in portfolio optimization. *Jobson* and *Korkie* (1980, 1981) and *DeMiguel* et al. (2009) show that simple equal-weighting actually outperforms mean-variance optimization in the presence of input parameter uncertainty, while *Best* and *Grauer* (1991) show how sensitive portfolio weights react to the estimated mean returns.

In practice, these problems are often circumvented by imposing additional constraints on individual assets' weights to generate more "intuitive" portfolios, resulting in theoretically inferior diversification. To improve upon this approach, several methods have emerged from academia to incorporate parameter uncertainty. In this paper, we will review the most common approaches to address the problem, including heuristic methods that encompass uncertainty by "resampling" and more sophisticated Bayesian methods that derive updated distributions of returns (*Rachev* et al., 2008). These methods have been shown to result in portfolios with less concentration, more stability and better out-of-sample performance.

For many investors however, these methods still lack a convenient way of expressing their subjective views. This has been addressed in seminal work by *Black* and *Litterman* (1990, 1991, 1992), where they develop the Black-Litterman model, an application of Bayesian statistics that allows an investor to specify subjective views on some assets. The views are combined with a market implied equilibrium to derive updated parameters for the vector of expected returns and the covariance matrix. It is a model with particular appeal for investors, as it provides a sophisticated statistical framework for practitioners, while allowing for discretion in specifying subjective views. Although the implementation of the Black-Litterman model is conceptually tractable, it hinges on the specification of a single scalar parameter of uncertainty; τ, the confidence an investor has in the market equilibrium. With complex effects of the parameter on the final allocation and a lack of clear guidance on how to set it, it poses a hurdle for the practical implementation of the model.

In *Fuhrer* and *Hock* (2023), a generalization of the Black-Litterman model with respect to the market equilibrium returns used to anchor the investors' views is proposed. More specifically, the single scalar confidence parameter τ of the original model is replaced with uncertainty parameters for each asset, introducing flexibility while still permitting the classical Black-Litterman model as a special case. As shown in *Fuhrer* and *Hock* (2023), this flexible model can then be used as the basis for a purely quantitative (and thus objective) parametrization of the equilibrium model, with favorable portfolio properties. While this approach is theoretically superior, implementation is demanding and might pose challenges to the model's practical use. Here, we therefore focus on a qualitative parametrization of the generalized model. We lend intuition to the uncertainty parameters on the one hand, and allow the investor to set them with a confidence interval approach on the other. In a simple four-asset example, we then illustrate the application of the generalized model to solve a classical asset allocation problem. By directly comparing the proposed allocations of both the classical Black-Litterman model and our more flexible model, we can highlight our model's practical usability.

The paper is structured as follows. In Section II, the literature on parameter uncertainty in portfolio optimization is reviewed. In Section III we briefly recap the classical Black-Litterman model and the generalization of *Fuhrer* and *Hock* (2023). Then, Section IV introduces qualitative approaches to parameterise the generalized model. Section V illustrates the use of this model and shows how the Black-Litterman model is encompassed as a special case, before Section VI concludes the paper.

II. Literature Review

The problem of estimation error of input parameters in the context of portfolio optimization is well documented: *Broadie* (1993) finds surprisingly large effects of estimation error. Due to the error maximization property of mean-variance analysis, he finds that the estimated performance of resulting portfolios are optimistically biased predictors of the actual portfolio performance, resulting in large misallocations. Similarly, *Chopra* (1993) shows that small errors in input parameters can lead to large changes in the composition of the optimal portfolio and *Best* and *Grauer* (1991) highlight how sensitive portfolio weights react to the estimated mean returns. This is further investigated by *Chopra* and *Ziemba* (1993). Using the concept of certainty equivalent loss (CEL), a monetary measure of the cost of picking the wrong portfolio, they show that for a given risk tolerance, the CEL for errors in means is approximately 10 times that for errors in variances and over 20 times that for errors in covariances. This resonates well with the results of *Merton* (1980) who uses a continuous-time model of stock prices and a simulation study to show that estimates of the mean, in contrast to estimates of volatility, would be very imprecise even with 30 years of data. While the volatility estimates could be improved by using higher-frequency data, estimates of the mean can only be improved by adding time periods, which is often not possible. *Kolm* et al. (2014) provide a more recent overview of the topic, reiterating the findings of *Jobson* and *Korkie* (1980, 1981) and *DeMiguel* et al. (2009) that show how naïve equal weighting of portfolio constituents actually outperforms mean variance optimization, and highlighting the economic impact of the problem.

Although not based on economic theory, heuristic approaches are commonly used by practitioners to deal with parameter uncertainty. First, constraints on individual portfolio weights should contain the problems associated with estimation error. The rationale behind these constraints is the observation by *Michaud* (1989) that mean-variance optimizers are "estimation-error maximizers": They overweight the securities or asset classes that have large estimated returns, negative correlations or small variances. But these are exactly the securities most likely to contain estimation error. *Frost* and *Savarino* (1988) discuss this approach in more detail, and *Jagannathan* and *Ma* (2003) show that imposing constraints on portfolio weights actually acts in the same way as shrinkage estimators for the covariance matrix do.

Second, the resampling approach, a technique proposed by *Michaud* (1998) and *Michaud* and *Michaud* (2008) has gained a lot of attraction by practitioners. It is intuitively better comprehensible than the statistically more sophisticated Bayesian approaches. An excellent review is provided by *Scherer* (2002). The resampling approach, as described in *Michaud* (1998), takes the same inputs as classical mean-variance optimization: A vector $\boldsymbol{\mu}$ of expected returns and a covariance matrix $\boldsymbol{\Sigma}$. It assumes returns to follow a multivariate normal distribution defined by these inputs. To encompass parameter uncertainty, the resampling approach then draws a large number of random samples from this multivariate normal distribution. For each of these random samples, a new $\tilde{\boldsymbol{\mu}}_s$ and $\tilde{\boldsymbol{\Sigma}}_s$ is obtained ($s = 1,...,S$, where S is the number of new samples). Each of these new parameter-pairs is then used as an input into a classical mean- variance optimization that can also have constraints on portfolio weights. As a result, S vectors of optimal portfolio weights $\tilde{\boldsymbol{\omega}}_s$ are obtained. The resampled optimal portfolio is then simply the mean vector over all S weight vectors. Portfolio resampling is thus, in essence, classical mean-variance optimization repeated a large number of times with slightly varying, simulated inputs. As pointed out by *Michaud* and *Michaud* (2008) and *Scherer* (2002), resampling has various appealing features. First, it produces portfolios that are better diversified and have a lower sensitivity to input parameters (less sudden shifts). Also, since the distribution of portfolio weights is available, estimation error is visualized and can be used, for instance, to implement a rebalancing approach. *Scherer* (2002) however notes that "there is no economic rationale derived" and points to other problems, especially in the case where no constraints on portfolio weights are present.

Besides these heuristic approaches, the Bayesian framework offers an alternative path to incorporate parameter uncertainty in the allocation process that is rooted in statistics. It allows to optimally combine two sets of information, usually sample and non-sample data (*Rachev* et al., 2008). A prior belief is updated with new data and optimally combined to the posterior distribution. As an intuition, Bayesian methods take into account that a view on one parameter of the model affects all other parameters as well[1]. For the problem of portfolio selection, Bayesian methods are used to derive updated posterior distributions of returns that incorporate parameter uncertainty. Of these, the most simple approach is the diffuse prior. It does not state any other view than that the parameters are estimated with some uncertainty. Most often, Jeffreys' prior (*Jeffreys* (1961)) is used to specify this very basic view. The resulting posterior distribution has the functional form of a multivariate normal distribution with the same mean as the sample data, but a scaled covariance matrix. The parameter uncertainty introduced through the diffuse prior thus leads to an overall increase in the perceived risk. As an alternative, a set of methods use informative (conjugate) priors, which allow the specification of prior distributions with well-defined properties (see *Frost* and *Savarino* (1986)). A particular application of conjugate priors are shrinkage estimators, as developed for instance in *Stein* (1956), *James* and *Stein* (1961), *Jorion* (1986) or *Ledoit* and *Wolf* (2003). In contrast, non-conjugate priors lead to posterior distributions that are not obtainable in an analytical form. As a result, views can be specified much more flexibly but applications rely on simulation methods, as, for instance, in *Markowitz* and *Usmen* (2005) or *Harvey* et al. (2008). Finally, *Meucci* (2008) proposes an approach where not even the prior belief is analytically expressed. There, it is possible to specify "fully flexible views in fully general non-normal markets". The approach relies on a methodology called entropy pooling, a generalization of Bayesian updating.

The Black-Litterman model is another application of the Bayesian framework to the problem of asset allocation[2]. As both the prior and posterior distributions are well-defined, it belongs to the same group of methods as the shrinkage estimators[3]. Indeed, it can be interpreted as a shrinkage method, where the investors' views act as the shrinkage target. *Kolm* et al. (2021) provide an overview of recent extensions to the original Black-Litterman model, including robust optimization and extensions to multiple-periods.

Several publications investigate the performance of the presented approaches to incorporate parameter uncertainty in the portfolio allocation process. *Wolf* (2006) finds both resampling and shrinkage estimators to outperform classical mean-variance optimization. *Markowitz* and *Usmen* (2005), *Fernandes* and *Ornelas* (2009), *Scherer* (2006) and *Harvey* et al. (2008) compare various Bayesian methods to resampling, with ambiguous results. *Fernandes* et al. (2012) propose

[1] For a through discussion of the Bayesian framework, consider *Rachev* et al. (2008) or *Scherer* (2010).

[2] Note that while the Black-Litterman model is considered to be a Bayesian method, *Avramov* and *Zhou* (2010) point to the fact that it is not entirely Bayesian, as the data generating process is not spelled out and the predictive density is not used.

[3] Consider A.1 for an alternative interpretation of the model in terms of Jeffreys' prior.

to combine the Black-Litterman model with resampling and actually show that this combined method outperforms in some cases. The largest study in this field is *Becker* et al. (2015), where seven different Bayesian estimators are compared to their resampled counterparts. They find the resampled versions to perform worse than the not-resampled equivalents.

III. Theory

In a first step, this section will develop the classical Black-Litterman model as published in *Black* and *Litterman* (1990, 1991, 1992) and *He* and *Litterman* (1999). In the interest of brevity, we only cover the theoretical steps required for understanding subsequent deviations. We follow *Fuhrer* and *Hock* (2023) closely and refer the interested reader there for additional details.

1. The Black-Litterman model

As an application of the Bayesian framework, the Black-Litterman model combines two sources of information: $\boldsymbol{\mu}$, the $(N \times 1)$ vector of expected excess returns derived from a market-based model (where N is the number of assets), and $\boldsymbol{q}$, a $(K \times 1)$ vector of subjective views on the assets' expected excess returns (where K is the number of views). Then, according to Bayes' rule, the following relationship holds:

$$f(\boldsymbol{\mu}|\boldsymbol{q}) \propto f(\boldsymbol{q}|\boldsymbol{\mu})\, f(\boldsymbol{\mu}). \tag{1}$$

In other words, in order to derive the posterior distribution of expected excess returns given the investors subjective views, the prior distribution of expected excess returns and an assumption about the distribution of the investors' views *given* the prior distribution are required. Bayes' theorem then allows to change the order of conditioning. To derive a prior distribution of expected excess returns, the classical Black-Litterman model derives the $(N \times 1)$ vector of equilibrium returns $\boldsymbol{\pi}$ from the *Sharpe* (1964) and *Lintner* (1965) CAPM:

$$\boldsymbol{\pi} = \boldsymbol{\beta}(R_M - R_f), \tag{2}$$

where $\boldsymbol{\beta}$ is an $(N \times 1)$ vector of market betas of the assets and $R_M - R_f$ is the market's excess return (above the risk free rate). To incorporate parameter uncertainty, the model assumes that the market is in equilibrium on average, but could be in disequilibrium at any given point in time. Therefore,

$$\boldsymbol{\mu} = \boldsymbol{\pi} + \epsilon \tag{3}$$

$$\text{with} \quad \epsilon \sim N(0, \boldsymbol{\Psi})$$

$$\text{and} \quad \boldsymbol{\Psi} = \tau\boldsymbol{\Sigma}, \tag{4}$$

where ϵ is a $(N \times 1)$ vector of random shocks that push the market off its long-run equilibrium and $\boldsymbol{\Psi}$ is the $(N \times N)$ covariance matrix of these shocks. Combining these yields the prior distribution of $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} \sim N(\boldsymbol{\pi}, \tau\boldsymbol{\Sigma}). \tag{5}$$

The scaling parameter τ simply scales the empirical covariance matrix and represents the uncertainty in the accuracy with which $\boldsymbol{\pi}$ is estimated. Given this prior distribution, it is now possible to specify subjective views on the absolute or relative performance of assets. They are specified with a $(K \times N)$ views matrix $\boldsymbol{P}$ of K views on N assets; the vector $\boldsymbol{q}$ of expected returns of the views; and a $(K \times K)$-covariance matrix $\boldsymbol{\Omega}$ of these views. $\boldsymbol{P}$ and $\boldsymbol{q}$ are of the following form:

$$\boldsymbol{P} = \begin{bmatrix} p_{1,1} & \cdots & p_{1,N} \\ \vdots & \ddots & \vdots \\ p_{K,1} & \cdots & p_{K,N} \end{bmatrix} \quad \boldsymbol{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_K \end{bmatrix},$$

where each line of $\boldsymbol{P}$ specifies a long only or long-short portfolio of the N assets and every element of $\boldsymbol{q}$ specifies the expected return of the respective portfolio. The returns of the views are also assumed to be expressed with uncertainty, such that:

$$\boldsymbol{q} = \boldsymbol{P}\boldsymbol{\mu} + \varepsilon \tag{6}$$

$$\text{where} \quad \varepsilon \sim N(0, \boldsymbol{\Omega}),$$

with ε a $(K \times 1)$ vector of random shocks. The original authors (*Black* and *Litterman*, 1990, 1991, 1992) propose a simplifying assumption for the covariance matrix of the shocks: The views are assumed to be uncorrelated, leading to a diagonal covariance matrix $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega} = \begin{bmatrix} \omega_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{k,k} \end{bmatrix}. \tag{7}$$

The distribution of the investors' views given the prior distribution of $\boldsymbol{\mu}$ is then the following multivariate normal distribution:

$$\boldsymbol{q} \mid \boldsymbol{\mu} \sim N(\boldsymbol{P\mu}, \boldsymbol{\Omega}). \tag{8}$$

Bayes' theorem can now directly be applied to combine the two sources of information consistently. The posterior distribution of expected excess returns given the investors subjective views is[4]

$$\boldsymbol{\mu} \mid \boldsymbol{q} \sim N(\boldsymbol{m}, \boldsymbol{V}), \tag{9}$$

where

$$\boldsymbol{m} = \boldsymbol{V}\left(\boldsymbol{\Psi}^{-1}\boldsymbol{\pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{q}\right) \tag{10}$$

and

$$\boldsymbol{V} = \left(\boldsymbol{\Psi}^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right)^{-1}. \tag{11}$$

This corresponds to the following multivariate normal distribution for the posterior distribution of the assets' returns:

$$\boldsymbol{R} \mid \boldsymbol{q} \sim N\left(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}\right) \tag{12}$$

$$\text{with} \quad \tilde{\boldsymbol{\mu}} = \boldsymbol{m} \quad \text{and} \quad \tilde{\boldsymbol{\Sigma}} = \Sigma + \boldsymbol{V}.$$

These updated parameters can now be used as the input to a portfolio optimization routine.

2. *Deviation from the Black-Litterman model*

In *Black* and *Litterman* (1992), the original authors reduce the problem of setting $\boldsymbol{\Psi}$ to setting a single scalar parameter τ, as they propose that $\boldsymbol{\Psi}$ is proportional to the covariance matrix $\boldsymbol{\Sigma}$ (see Equation (4)). However, as discussed in *Fuhrer* and *Hock* (2023), this assumption induces rigidity in the model and makes it opaque, as there are no guidelines on how to set the value of τ, or even how to interpret it intuitively. This poses a challenge for practical applications.

Thus, *Fuhrer* and *Hock* (2023) propose a simple deviation of the Black-Litterman model to allow a more flexible specification of $\boldsymbol{\Psi}$, while still encompassing the Black–Litterman specification as a special case. Here, we briefly reproduce

[4] Consider for instance *Satchell* and *Scowcroft* (2007) for a mathematical proof.

that deviation for the reader's convenience, but refer to the original paper for more details.

In a first step, the covariance matrix of the returns $\mathbf{\Sigma}$ is decomposed into a volatility vector $\boldsymbol{\sigma}$ and a correlation matrix $\mathbf{\Phi}$:

$$\mathbf{\Sigma} = \operatorname{diag}(\boldsymbol{\sigma})\mathbf{\Phi}\operatorname{diag}(\boldsymbol{\sigma}). \tag{13}$$

The correlation matrix captures information about the co-movement of equilibrium returns, but a new vector of standard errors of estimated equilibrium returns $\hat{\boldsymbol{\sigma}}$ can be chosen by the investor. Then $\mathbf{\Psi}$ simply combines these two:

$$\mathbf{\Psi} = \operatorname{diag}(\hat{\boldsymbol{\sigma}})\mathbf{\Phi}\operatorname{diag}(\hat{\boldsymbol{\sigma}}). \tag{14}$$

If the investor sets $\hat{\boldsymbol{\sigma}} = \sqrt{\tau}\boldsymbol{\sigma}$ (i. e. the standard errors of estimated equilibrium returns are proportional to the empirical volatilities of returns) the classical Black-Litterman model can be recovered:

$$\mathbf{\Psi} = \operatorname{diag}(\hat{\boldsymbol{\sigma}})\mathbf{\Phi}\operatorname{diag}(\hat{\boldsymbol{\sigma}}) = \operatorname{diag}\left(\sqrt{\tau}\boldsymbol{\sigma}\right)\mathbf{\Phi}\operatorname{diag}\left(\sqrt{\tau}\boldsymbol{\sigma}\right) = \tau\,\Sigma. \tag{15}$$

Using this decomposition, *Fuhrer* and *Hock* (2023) develop a multi-stage econometric procedure to specify uncertainty about equilibrium returns for each asset individually, which utilizes information from underlying index constituents to solve an asset allocation problem. In contrast, this study introduces a method that addresses heuristic or qualitative descriptions of uncertainty – provided, for instance, by a portfolio manager or the investment committee of a pension fund – and offers practical guidance for application in investment practice.

IV. Intuition through Judgemental Approach

In order to increase the intuition in specifying the uncertainty in the expected equilibrium returns, we follow and generalise the Judgemental Approach proposed in *Rachev* et al. (2008) and *Scherer* (2010) to specify uncertainty in the expected equilibrium returns. What follows is our proposition to model this uncertainty flexibly and consistently, and to integrate it into the Black-Litterman model.

1. Symmetric Confidence Intervals

First, consider the case of symmetric confidence intervals. For each asset, the parameters of interest are π and σ_π. The relationship of the confidence level α, the upper and lower limits u_π and l_π and the parameters of interest is depicted in Figure 1.

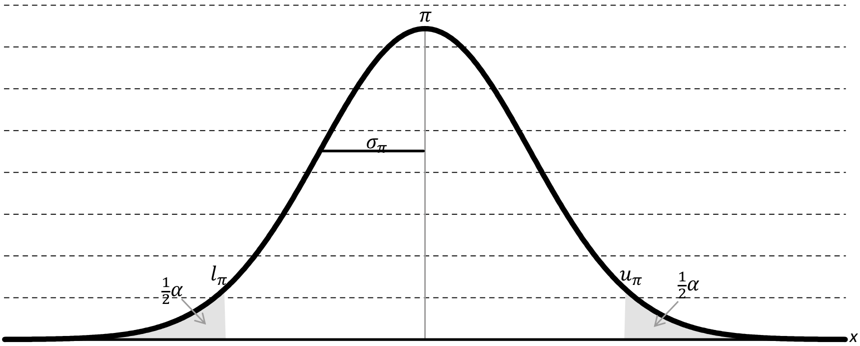

Figure 1: Symmetric confidence intervals of a normal distribution

Note: π is the expected return, σ_π the standard deviation of the expected return, α the confidence level (for a 95 % confidence level, $\alpha = 5\,\%$) and l_π and u_π are the lower and upper limit, respectively.

There are two fundamental relations between the parameters:

(16) $$\pi = \frac{u_\pi + l_\pi}{2}$$

and

(17) $$\sigma_\pi = \frac{l_\pi - \pi}{Z_{\alpha/2}},$$

where $Z_{\alpha/2}$ is the $(\alpha / 2)$-quantile of the standard normal distribution.

Table 1

Different parameter-combinations elicited form the investor to compute π and σ_π

User-specified	π	σ_π
π, l_π, α	π	$\frac{l_\pi - \pi}{Z_{\alpha/2}}$
π, u_π, α	π	$\frac{\pi - u_\pi}{Z_{\alpha/2}}$
l_π, u_π, α	$\frac{u_\pi + l_\pi}{2}$	$\frac{l_\pi - u_\pi}{Z_{\alpha/2}}$

With these, the investor can be asked to specify various combinations of parameters to arrive at π and σ_π. The most convenient ones are tabulated in Table 1. In the first and second line of Table 1 the investor specifies his expectations about the mean return, and either a lower or an upper limit that the mean return is expected to lie in with a confidence level $(1-\alpha)$. The last line of Table 1 is probably the most intuitive: The investor simply states the interval the expected equilibrium return lies in, with a confidence in that interval.

2. Asymmetric Confidence Intervals

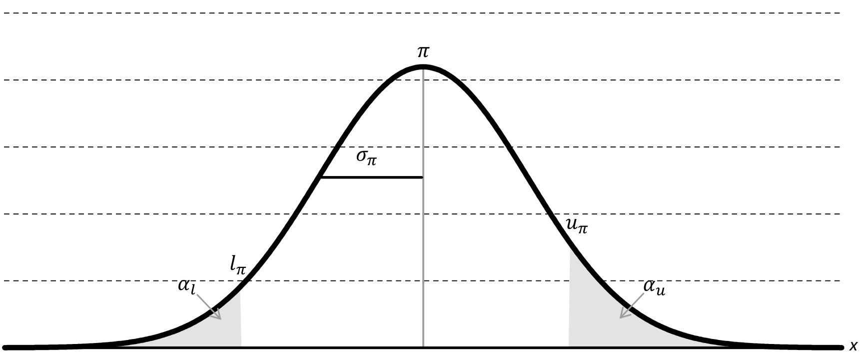

Figure 2: Asymmetric confidence intervals of a normal distribution

Note: Asymmetric confidence intervals of a normal distribution. Notation is as in Figure 1, with the exception that α is allowed to be asymmetric $(\alpha = \alpha_l + \alpha_u$, with $\alpha_l \neq \alpha_u)$. The investor can specify the probability mass in the tails independently.

Since the expected mean returns are required to follow a normal distribution, introducing asymmetry in the estimate is not possible in this framework. It is, however, possible for the investor to specify an asymmetric confidence interval. The situation is depicted in Figure 2 and is very similar to the situation in Figure 1 with symmetric confidence intervals. But here, the probability mass in the tails (α_l and α_u) can be specified independently. Then,

$$\pi = \frac{u_\pi Z_{\alpha_l} + l_\pi Z_{1-\alpha_u}}{Z_{\alpha_l} - Z_{1-\alpha_u}} \tag{18}$$

and

$$\sigma_\pi = \frac{l_\pi - \pi}{Z_{\alpha_l}}. \tag{19}$$

3. *Correspondence to τ*

The uncertainty parameter τ in the classical Black-Litterman model is simply the parameter of proportionality between $\mathbf{\Psi}$ and $\mathbf{\Sigma}$. Given a specification of $\boldsymbol{\sigma}_\pi$, it is possible to recover the implied τ_k for each asset k. Take the empirical variance of the returns σ_k^2 and the specified variance of the estimate of the equilibrium return, $\sigma_{\pi,k}^2$, to compute:

$$\tau_k = \frac{\sigma_{\pi,k}^2}{\sigma_k^2}. \tag{20}$$

Note that we index τ over k as well, as with the added flexibility, τ is specific to every asset. This correspondence then allows the computation of implicit confidence intervals from the assumptions of the Black-Litterman model. Given $\boldsymbol{\pi}$, $\mathbf{\Sigma}$ and τ from the model, and assuming a confidence level of α, the corresponding symmetric confidence intervals can readily be obtained as:

$$[\boldsymbol{l}_\pi, \boldsymbol{u}_\pi] = \boldsymbol{\pi} \mp \tau \cdot \mathrm{diag}(\mathbf{\Sigma}) \cdot Z_{\alpha/2}. \tag{21}$$

For asymmetric confidence intervals, with α_l and α_u specified by the investor, the interval is defined as:

$$\boldsymbol{l}_\pi = \boldsymbol{\pi} - \tau \cdot \mathrm{diag}(\mathbf{\Sigma}) \cdot Z_{\alpha_l} \tag{22}$$

and

$$\boldsymbol{u}_\pi = \boldsymbol{\pi} + \tau \cdot \mathrm{diag}(\mathbf{\Sigma}) \cdot Z_{1-\alpha_u}. \tag{23}$$

We propose to use these intervals as a starting point for the investor. From there, she can adjust the intervals and confidence according to her knowledge about the models generating the estimated equilibrium returns.

V. Illustration

To illustrate how the generalised model can be used to derive the updated input parameters for portfolio optimization and the resulting asset allocation, a simple four-asset example will be introduced in this section. The section is structured as follows: First, the data and expected equilibrium returns are presented. Then, the classical Black-Litterman model with a constant τ but without views is implemented. The resulting allocation serves as the basis of comparison. We then introduce our flexible model, by allowing the investor to specify uncertainty in the equilibrium model other than through the constant τ. Next, we introduce a set of subjective views and recompute both the Black-Litterman model and our flexible model under these views. This allows to identify and discuss the effects of our results.

1. A Four-Asset Example

Consider a simple example with four asset classes to choose from: Global equities (GE), global government bonds (GGB), emerging market bonds (EMB) and real estate funds (REF)[5]. Assume that they are characterised by the following historical mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ of risk premia:

	$\boldsymbol{\mu}$
GE	6.43 %
GGB	3.26 %
EMB	4.76 %
REF	9.06 %

$\boldsymbol{\Sigma}$	**GE**	**GGB**	**EMB**	**REF**
GE	1.78 %	–0.16 %	1.04 %	1.31 %
GGB	–0.16 %	0.23 %	–0.13 %	0.00 %
EMB	1.04 %	–0.13 %	1.40 %	0.89 %
REF	1.31 %	0.00 %	0.89 %	3.24 %

As suggested in Section 3.2, $\boldsymbol{\Sigma}$ can be decomposed into historical volatilities $\boldsymbol{\sigma}$ and the correlation matrix $\boldsymbol{\Phi}$:

[5] Consider Appendix A.2 for details on the time series used.

	σ
GE	13.35 %
GGB	4.74 %
EMB	11.85 %
REF	18.00 %

Φ	GE	GGB	EMB	REF
GE	1.00	–0.25	0.66	0.54
GGB	–0.25	1.00	–0.23	0.00
EMB	0.66	–0.23	1.00	0.42
REF	0.54	0.00	0.42	1.00

Suppose that the following equilibrium returns are derived from an equilibrium model (for instance, the CAPM):

	π
GE	3.50 %
GGB	0.60 %
EMB	2.50 %
REF	3.00 %

2. Asset Allocation without Views

In a first step, we illustrate how the added flexibility in the proposed model influences the asset allocation when the investor has no subjective views about absolute or relative performances of assets. As a basis of comparison, we first derive the asset allocation using the Black-Litterman model with the constant τ set to 0.05 in Section 5.2.1. Implied confidence intervals are derived and presented. Then, in Section 5.2.2, we will slightly adjust these confidence intervals and, using the flexible model, will derive a new asset allocation. A comparison of the two allocations allows the identification of the effects of the flexible model specification.

Using the Black-Litterman Model

To use the Black-Litterman model without investor's subjective views, only the hyperparameter τ has to be set. We choose to set $\tau = 0.05$, which is in line with suggestions summarized in *Idzorek* (2007). According to Equation (4), the uncertainty about the expected equilibrium returns $\boldsymbol{\Psi}$ is simply the scaled covariance matrix $\boldsymbol{\Psi} = \tau \boldsymbol{\Sigma}$:

Ψ	GE	GGB	EMB	REF
GE	0.089 %	–0.008 %	0.052 %	0.065 %
GGB	–0.008 %	0.011 %	–0.007 %	0.000 %
EMB	0.052 %	–0.007 %	0.070 %	0.045 %
REF	0.065 %	0.000 %	0.045 %	0.162 %

For illustrative purposes, and following Section 3.2, we decompose $\mathbf{\Psi}$ into the volatility vector of the expected equilibrium returns $\boldsymbol{\sigma}_{\pi}$ and the correlation matrix $\mathbf{\Phi}$:

	$\boldsymbol{\sigma}$
GE	2.99 %
GGB	1.06 %
EMB	2.65 %
REF	4.03 %

$\mathbf{\Phi}$	GE	GGB	EMB	REF
GE	1.00	–0.25	0.66	0.54
GGB	–0.25	1.00	–0.23	0.00
EMB	0.66	–0.23	1.00	0.42
REF	0.54	0.00	0.42	1.00

As expected, the correlation matrix $\mathbf{\Phi}$ is equivalent to the correlation matrix of the data presented in Section 5.1 and the volatility vector of the expected equilibrium returns is simply the scaled volatility vector of the returns $\boldsymbol{\sigma}_{\pi} = \sqrt{\tau}\boldsymbol{\sigma}$. It is also possible to obtain the implied symmetric confidence intervals of this specification for an assumed confidence level of $(1 - \alpha) = 80\,\%$ (i. e. 10 % of probability mass in each symmetric tail), derived using Equation (21):

	π	l_k^{π}	u_k^{π}	$\alpha_{l,k}$	$\alpha_{u,k}$	τ
GE	3.50 %	–0.33 %	7.33 %	10 %	10 %	0.050
GGB	0.60 %	–0.76 %	1.96 %	10 %	10 %	0.050
EMB	2.50 %	–0.90 %	5.90 %	10 %	10 %	0.050
REF	3.50 %	–1.66 %	8.16 %	10 %	10 %	0.050

All the inputs required to compute the updated parameters $\tilde{\boldsymbol{\mu}}$ and $\tilde{\Sigma}$ from the Black-Litterman model are now defined. Using Equations (10)–(12), the following updated parameters are obtained:

	$\tilde{\mu}$
GE	3.50 %
GGB	0.60 %
EMB	2.50 %
REF	3.50 %

$\tilde{\Sigma}$	**GE**	**GGB**	**EMB**	**REF**
GE	1.87 %	–0.16 %	1.09 %	1.37 %
GGB	–0.16 %	0.24 %	–0.14 %	0.00 %
EMB	1.09 %	–0.14 %	1.47 %	0.93 %
REF	1.37 %	0.00 %	0.93 %	3.40 %

	$\tilde{\sigma}$
GE	13.68 %
GGB	4.86 %
EMB	12.14 %
REF	18.45 %

These parameters can be used as the inputs of a classical mean-variance optimization problem as proposed by *Markowitz* (1952). The resulting portfolio weights $\boldsymbol{\omega}_{BL}$ are reported below. The weights are constraint to add to one, the portfolio volatility σ^p is constraint to 8.00 % and the expected equilibrium return is maximised.

	$\boldsymbol{\omega}_{BL}$
GE	46.61 %
GGB	35.18 %
EMB	8.85 %
REF	9.35 %

Using the Flexible Model

In the previous section, by setting the hyperparameter τ, the investor implicitly specified the same level of confidence in each of the estimated equilibrium returns. This becomes obvious when investigating the implied confidence intervals already derived in Section 5.2.1 above, where τ is constant for all assets.

Suppose, however, that the investor derives these estimates not from a single equilibrium model, but from specific models for each asset, or that experience suggests that equilibrium returns for some assets are more reliably estimated than for others. For instance, the investor might have a smaller confidence in the estimate of the equilibrium return of global equities. We represent this by leaving the upper and lower limit unchanged, but change the mass in the tails from 10 % to 20 % for each. Our confidence in the estimated equilibrium returns thus decreases from 80 % to 60 %. The second modification we propose is that the investor has a very specific equilibrium model for emerging market bonds, which suggests a higher confidence in the form of limits closer to the mean and a slightly asymmetric confidence interval[6]. Both modifications are added in the table below:

	$\boldsymbol{\pi}$	$\boldsymbol{l_k^{\pi}}$	$\boldsymbol{u_k^{\pi}}$	$\boldsymbol{\alpha_{l,k}}$	$\boldsymbol{\alpha_{u,k}}$	$\boldsymbol{\tau}$
GE	3.50 %	–0.33 %	7.33 %	20 %	20 %	0.116
GGB	0.60 %	–0.76 %	1.96 %	10 %	10 %	0.050
EMB	2.50 %	1.00 %	4.50 %	11 %	5 %	0.011
REF	3.50 %	–1.66 %	8.16 %	10 %	10 %	0.050

The parameter τ is no longer constant in this case. It is higher for global equities, representing the higher uncertainty in the equilibrium estimate, and lower for emerging market bonds, as the investor is able to estimate the equilibrium returns more accurately. On average, τ is still very close to 0.05, guaranteeing that results are not driven by additional risk of the equilibrium model.

The new assumptions about the expected equilibrium returns result in the following parameters $\boldsymbol{\sigma}_{\pi}$ and $\boldsymbol{\Psi}$:

	$\boldsymbol{\sigma}_{\pi}$
GE	4.55 %
GGB	1.06 %
EMB	1.22 %
REF	4.03 %

$\boldsymbol{\Psi}$	**GE**	**GGB**	**EMB**	**REF**
GE	0.207 %	–0.012 %	0.036 %	0.100 %
GGB	–0.012 %	0.011 %	–0.003 %	0.000 %
EMB	0.036 %	–0.003 %	0.015 %	0.020 %
REF	0.100 %	0.000 %	0.020 %	0.162 %

Using Equations (10)–(12) also here, the updated parameters are obtained as follows:

	$\tilde{\boldsymbol{\mu}}$
GE	3.50 %
GGB	0.60 %
EMB	2.50 %
REF	3.50 %

$\tilde{\boldsymbol{\Sigma}}$	**GE**	**GGB**	**EMB**	**REF**
GE	1.99 %	–0.17 %	1.08 %	1.41 %
GGB	–0.17 %	0.24 %	–0.13 %	0.00 %
EMB	1.08 %	–0.13 %	1.42 %	0.91 %
REF	1.41 %	0.00 %	0.91 %	3.40 %

	$\tilde{\boldsymbol{\sigma}}$
GE	14.10 %
GGB	4.86 %
EMB	11.91 %
REF	18.45 %

[6] To insure that the expected equilibrium return of emerging market bonds is unchanged at 2.50 %, the probability mass in the lower tail has the somewhat odd value of 11 %.

With the exact same optimization routine used above (σ^p = 8.00 %), the new weights are obtained and compared to the Black–Litterman solution below:

	ω_{BL}	ω_{FL}
GE	46.61 %	41.34 %
GGB	35.18 %	34.50 %
EMB	8.85 %	13.81 %
REF	9.35 %	10.35 %

As would be expected, the added uncertainty about the equilibrium returns of global equities leads to a smaller proportion of wealth invested in this asset class. Conversely, a larger proportion of wealth is invested in emerging market bonds, as the expected equilibrium return is specified with higher confidence. Differences in the weights of the other asset classes stem from the propagation of the effects through the correlation matrix with the Bayesian methodology inherent to the Black–Litterman approach.

3. *Investor's Subjective Views*

In a second comparison, investor's subjective market views are introduced. They are defined as in the original Black-Litterman model. It is assumed that the investor holds three distinct views about the four assets:

1. Global equities (GE) will have a performance of 4 % for the foreseeable future.
2. Global government bonds (GGB), on the other hand, have an expected return of only 0.5 %.
3. Real estate funds (REF) will outperform global government bonds (GGB) and emerging market bonds (EMB) by 1 percentage point.

From this, both $\boldsymbol{P}$ and $\boldsymbol{q}$ can be obtained:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \quad \boldsymbol{q} = \begin{bmatrix} 4.0\% \\ 0.5\% \\ 1.0\% \end{bmatrix}.$$

In order to specify the uncertainty about these views, the method proposed by *He and Litterman* (1999) is used. Accordingly, the diagonal elements of $\boldsymbol{\Omega}$ can be determined as $\omega_{k,k} = \tau \boldsymbol{p}_k \boldsymbol{\Sigma} \boldsymbol{p}_k^{'}$ and the off-diagonal elements are assumed to be zero. We still assume a constant τ = 0.05 for the views, yielding the following views-covariance matrix:

Ω	View 1	View 2	View 3
View 1	0.09 %	0	0
View 2	0	0.01 %	0
View 3	0	0	0.13 %

It is important to note that these views are used identically in both the classical Black-Litterman model illustration, as well as in the illustration of the more general model. This allows to isolate the impact of our modification on the asset allocation when views are present.

4. Asset Allocation with Views

The views just introduced will now be implemented in both the classical Black-Litterman model as well as in the more flexible model. To allow a comparison, the setup in both models is kept equivalent to the analysis in Section 5.2, and the same views, as just outlined, are introduced to both models.

Using the Black-Litterman Model

As before, we use Equations (10) – (12) to compute the updated parameters from the Black-Litterman model, this time taking into account the investor's subjective views:

	$\tilde{\mu}$
GE	3.67 %
GGB	0.54 %
EMB	2.66 %
REF	3.16 %

$\tilde{\Sigma}$	**GE**	**GGB**	**EMB**	**REF**
GE	1.82 %	–0.16 %	1.07 %	1.33 %
GGB	–0.16 %	0.23 %	–0.13 %	0.00 %
EMB	1.07 %	–0.13 %	1.46 %	0.92 %
REF	1.33 %	0.00 %	0.92 %	3.32 %

	$\tilde{\sigma}$
GE	13.51 %
GGB	4.80 %
EMB	12.07 %
REF	18.22 %

The parameter $\tilde{\mu}$ is pulled into the direction of the views. Especially View 3 has a non-trivial influence on the parameter: As it assumes the relative return of global government bonds and emerging market bonds to real estate funds to be smaller than it actually is, the expected return of real estate funds is reduced, while that of global government bonds and emerging market bonds is increased. The effect on global government bonds is however not visible, as there is an interaction with the absolute View 2, targeted directly on this asset class. The resulting volatilities are generally smaller, as the added information reduces the uncertainty in the model.

The resulting asset allocation is reported below and compared to the Black-Litterman model without views. As can be seen, the allocation weights are pulled into the direction indicated by the views.

	$\boldsymbol{\omega}_{BL}$	$\boldsymbol{\omega}_{BL}^{V}$
GE	46.61 %	51.70 %
GGB	35.18 %	33.96 %
EMB	8.85 %	12.08 %
REF	9.35 %	2.25 %

Using the Flexible Model

The same experiment is repeated for the more flexible model. All specifications are equivalent to the specifications in Section 5.2.2, but the views as outlined in Section 5.3 are introduced. The resulting parameters are the following:

	$\tilde{\boldsymbol{\mu}}$
GE	3.75 %
GGB	0.54 %
EMB	2.55 %
REF	3.13 %

$\tilde{\boldsymbol{\Sigma}}$	**GE**	**GGB**	**EMB**	**REF**
GE	1.84 %	–0.16 %	1.05 %	1.32 %
GGB	–0.16 %	0.23 %	–0.13 %	0.00 %
EMB	1.05 %	–0.13 %	1.41 %	0.90 %
REF	1.32 %	0.00 %	0.90 %	3.31 %

	$\tilde{\boldsymbol{\sigma}}$
GE	13.57 %
GGB	4.80 %
EMB	11.89 %
REF	18.19 %

First, note how the additional uncertainty in the equilibrium returns of global equities is propagated to the volatility vector $\tilde{\boldsymbol{\sigma}}$ used in the optimization, and conversely for emerging market bonds. Second, the impact on $\tilde{\boldsymbol{\mu}}$ is more complex in this case: The expected return of global equities is increased because the added uncertainty about the equilibrium return pulls the model more toward View 1, which assumes a higher return of global equities. For emerging market bonds, where the expected equilibrium return is estimated with less uncertainty, the return is pulled more towards that equilibrium return than the return implied by View 3.

The resulting allocation is reported in the next table, where a comparison to the results of the more flexible model without views is provided as well.

	ω_{FL}	ω_{FL}^{V}
GE	41.34 %	54.89 %
GGB	34.50 %	35.69 %
EMB	13.81 %	7.87 %
REF	10.35 %	1.55 %

The overall results are illustrated graphically in Figure 3.[7] The effects of accurately reflecting the uncertainty about expected equilibrium returns in the asset allocation framework are visible, as the more flexible model clearly chooses different weights than the classical Black-Litterman model. Without views, more (un)certainty about expected equilibrium returns tends to increase (decrease) the weight of the respective asset. When views are introduced, effects are more complex, as the uncertainty about expected equilibrium returns is traded off with the uncertainty in the views, and thus an interaction of effects is taken into account. This also shows the advantages of using the more flexible model: These effects are accounted for consistently in a Bayesian framework, presenting the investor with a flexible tool for the asset allocation process.

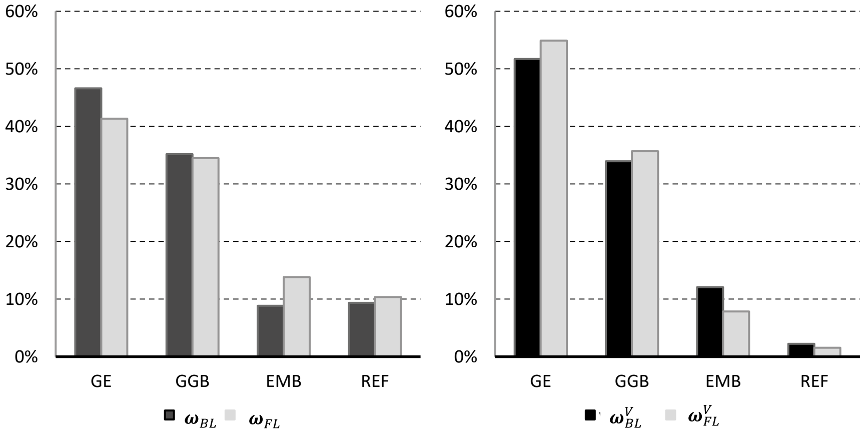

Figure 3: Optimized portfolio weights for the Black-Litterman model and the more flexible model, with and without views

Note: Optimized portfolio weights with and without views, based on a portfolio volatility target of 8 %.

[7] All results are based on a portfolio volatility target of 8 %. Please refer to Appendix A.3 for portfolio weights corresponding to different volatility targets.

VI. Conclusion

To attenuate one of the major problems of mean-variance optimization, several approaches to incorporate parameter uncertainty into the asset allocation process have been proposed in the academic literature. Of these, the Black-Litterman model has practical appeal for institutional investors, as it also allows the investor to specify subjective views that are consistently incorporated into the allocation using sophisticated Bayesian methods. The specification of the uncertainty parameter τ, controlling the degree of uncertainty about the equilibrium returns, is however not trivial and introduces rigidity. A more flexible version of the model is introduced in *Fuhrer* and *Hock* (2023), where an entirely data-driven approach is used to flexibly parameterise the model. We extend this method with a parametrization of the flexible model based on a more intuitive, judgmental approach that can also be reversed, i. e. used to lend intuition to the specification of τ in the classical Black-Litterman model. Our qualitative approach is illustrated in a simple four-asset example of a classical asset allocation problem. The results clearly indicate that our proposition has an influence on the resulting asset allocation, and how these effects are propagated through the Bayesian nature of the model.

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VII. A. Appendix

A.1. Alternative Interpretation of τ

There is an alternative interpretation of τ available when comparing the results of Black– Litterman to the results of diffuse priors from Bayesian statistics. The use of diffuse or uninformative priors is an established methodology to incorporate estimation risk in the allocation process (see, for instance *Rachev* et al. (2008)). Using Jeffreys' prior (*Jeffreys* (1961)), it can be shown that the posterior distribution of returns has the following parameters:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} \quad \textit{and} \quad \tilde{\boldsymbol{\Sigma}} = \frac{\left(1+\frac{1}{T}\right)(T-1)}{T-N-2}\boldsymbol{\Sigma}. \tag{A.1}$$

To compare this to the Black-Litterman assumption above, consider the Black-Litterman model with no views. In this case,

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{m} = \boldsymbol{\pi} \quad \textit{and} \quad \tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + \mathbf{V} = \boldsymbol{\Sigma} + \boldsymbol{\Psi} = (1+\tau)\boldsymbol{\Sigma}. \tag{A.2}$$

This allows us to see the connection between using Jeffreys' prior and the Black-Litterman model without subjective views and lends an intuitive interpretation for the parameter τ. In the example above, with $N = 4$ assets, τ becomes a function of T, the number of data points used to estimate the expected returns in the method using Jeffreys' prior:

$$\tau = \frac{\left(1+\frac{1}{T}\right)(T-1)}{T-6} - 1. \tag{A.3}$$

For instance, setting $T = 36$ (i.e. we estimate expected returns from three years of monthly data), then $\tau = 0.2$. If $T = 120$ (i.e. ten years of data), $\tau = 0.05$.

A.2. Construction of Time Series

For the illustration, we use the following time series (all data obtained from Bloomberg):

	Time Series
GE	MSCI World Total Return (hedged to CHF) minus 3M Libor CHF
GGB	50 % FTSE US GBI (constant Duration 6.5 Year) minus 3M US Govt. Bond Yield
	50 % FTSE Germany GBI (constant Duration 6.5 Year) minus 3M Libor EUR
EMB	Bloomberg Barclays EM USD Aggregate Total Return unhedged minus
	FTSE US GBI (Duration matched)
REF	FTSE NAREIT Composite Total Return minus 3M US Govt. Bond Yield

1. A.3. Portfolio Weights for Different Target Risk Levels

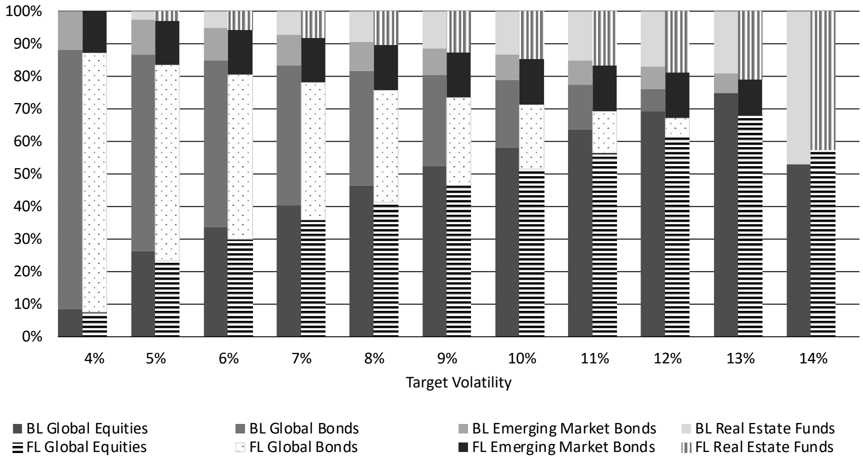

Figure 4: Allocation Weights for different Target Volatilities (without Views)

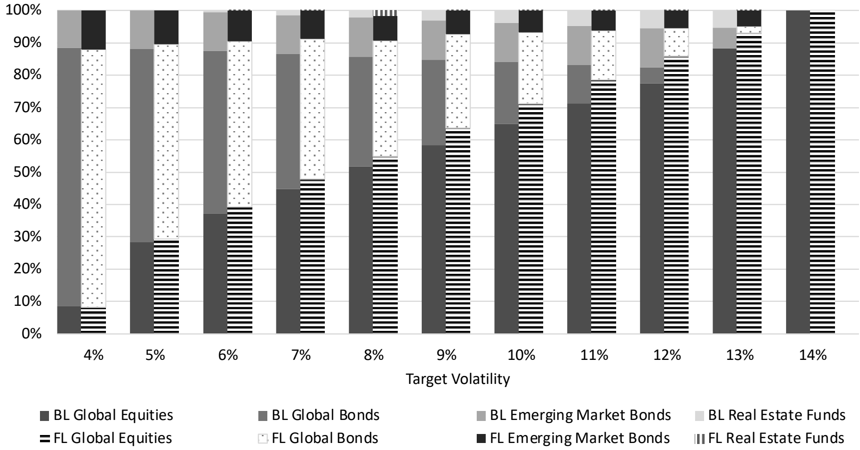

Figure 5: Allocation Weights for different Target Volatilities (with Views)