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Catastrophic Risk and Egalitarian Principles for Risk Transfer Mechanisms

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Abstract

Financial aid for the worst off victims of earthquakes and other catastrophes seems tobe a morally unquestioned principle for the allocation of public funds. This paper shows however, that this principle is ambiguous if the decision is viewed as a dynamic choice problem where such resources need to be allocated in two periods: before and after the event takes place (before and after uncertainty is resolved). The literature on social choice suggests that utilitarian principles fare better in such situations. This paper provides a uniform formal framework to relate one such result, namely a multi-profile version of Harsanyis 1955 theorem by Mongin (1994) to another one by Myerson (1981), stated in a somewhat unconventional social choice framework. It shows that the Linearity condition, that is met only by welfare functions of the utilitarian type, has a natural interpretation in terms of an equivalence of ex ante and ex post evaluation, a concept that is related to but not equivalent with dynamic consistency.

JEL Classification: G22, H42, I3, Q54

1. Introduction

When catastrophic events occur, it seems to be a morally unquestioned principle for the allocation of public funds that the worst off victims get compensation for their losses. This principle is ambiguous if the decision to provide relief for the victims is regarded as a dynamic choice problem where egalitarian policies in favour of the worst off need to be carried out in two periods: before and after the event takes place (before and after uncertainty is resolved). A policy that concentrates on preventing harm ex ante for the worst-off considering expected losses may no longer be justified ex post, since other victims may turn out to be worst off then.

It has often been stated that utilitarian policies do not have the same problems to be coherently implemented over time as egalitarian ones. One of the best known arguments, that could already be interpreted in this direction is Harsanyi's 1955 aggregation theorem, that basically states the following:

Given von Neuman-Morgenstern (VNM) individual utility functions and a VNM social welfare function and some Pareto condition, utilitarianism is implied. Whilst the formal argument cannot be questioned, the ethical relevance of the result continues to be debated, for a concise discussion see Mongin / d'Aspremont (1999). However, Myerson (1981) explicitly contrasts Utilitarianism and Egalitarianism in a somewhat unconventional social choice framework and shows that the Linearity condition, that is met only by welfare functions of the utilitarian type, has a natural interpretation in terms of an equivalence of ex ante and ex post evaluation, a concept that is related to but not equivalent with the concept of dynamic consistency. Hammond (1981 and 1983) discusses the question in a richer temporal framework and concludes that ex post welfare optimality, which is compatible with dynamic consistency, should be the ethically relevant criterion.

Together with a numeric example illustrating the difficulty for egalitarian policies in a dynamic setting, this paper provides a uniform formal framework to state a result that relates a multi-profile version of Harsanyis theorem by Mongin (1994) to the findings of Myerson (1981) and concludes with some lessons for dynamically consistent policies that want to meet egalitarian concerns before and after catastrophic events.

We should justify that we even consider expected utility theory to be informative in a context of catastrophic risk. Not even Oskar Morgenstern (1979) thought that the VNM expected utility theory he initiated with John von Neumann could be useful for low probability events. However, the concern that the multiplicative and additive form of the VNM expected utility functions invite to disregard very low probability high consequence events is often neglected. I believe that this is unfortunate and that for such situations the search for a viable normative theory should continue. But still, dynamic consistency should figure among the properties such an alternative theory exhibits. From dynamic decision theory in the individual context we know how intimately linked dynamic consistency and the independence axiom of expected utility theory are, even though independence can be weakened while keeping dynamic consistency. See e.g. the careful exposition of this question by Cubitt (1996). This is why we think, that the study of dynamic consistency, the independence axiom and its consequences for egalitarian risk policies, i.e. policies that favour the worst off victims is also fruitful in a collective choice context, where dynamic consistency as we will see is per se an ambiguous concept.

2. A Numeric Example

Let us consider the following numeric example[1]: Suppose a small society with only two individuals, named A and B, and let us have three periods. The society faces a significant risk of some natural or man made disaster in period 2, where the probability of the disaster is denoted by p_d, whilst p_{nd} denotes the probability for the absence of the event, and $p_d = 1 - p_{nd}$. You can find the associated decision tree in Figure 1, where the respective outcomes after period 3 are given as vectors of utilities $(U_A, U_B) \in \mathbb{R}^2$ where U_A denotes the outcome for individual A and U_B denotes the outcome for individual B, after all taxes and transfers are considered. In general, the two individuals will have their subjective estimates about how probable the event is, but for the time being, we will take the social decision maker's probability evaluation as given and for convenience we will assume that $d = 0,05$. Whilst in period 2 nature has its move (represented in the decision tree in figure 1 by circle nodes) decisions of the society need to be taken in period 1 and 3, represented by squares in the diagram. The decision to be taken in period one concerns e.g. two different building regulations, or two different public programmes to increase insurance coverage x and y that both affect the individuals in a different way and that have an impact on how well off the individuals are in case of the disaster, say e.g. a flooding, an earthquake or a terrorist attack.

However, in period 3 the society needs to make a decision whether to adopt a policy x' that involves income equalising transfers. In case of the disaster these transfers can be interpreted as a relief-programme for that individual that is worse off. If the disaster did not take place, the transfers can be thought of as related to help finance some even stronger building improvement programme, to better prepare for the next flood, or whatsoever.[2]

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1 A similar example is given in Fleurbaey (1996), I wish to thank an anonymous referee for the idea to adapt it to better suit disaster risk in terms of probability.

2 It may seem awkward to allocate resources for preventive measures after a disaster. In order to justify this we either should include some future periods, which would make the example more complicated. The second option is, as we do here, to assume that the society perceived well that they just had the luck to not undergo a major disaster. In this case, we may well accept a general reasoning of the decision maker as a further constraint: "After disaster is before the next disaster and some measures will be accepted only in case of a disaster or the dramatic perception that disasters can hit our society, this is why the set of feasible preventive options is larger after the "disaster chance node" has been resolved and thus it is rational to implement some of those preventive measures not feasible in period 1". This kind of constraints and reasoning should well be accommodated when determining dynamically coherent public policy choice.

However, the interpretation of the numbers should not be exaggerated, since they include the cumulative effect of all the measures taken, that are not fully specified and we also do not know the initial utility levels of the individuals in this example without any policy and / or disaster. One individual could also be the issuer of insurance, the premium of which might get subventions financed by general taxation in some of the policy plans, but also the transfers could be financed by cat bonds placed outside the

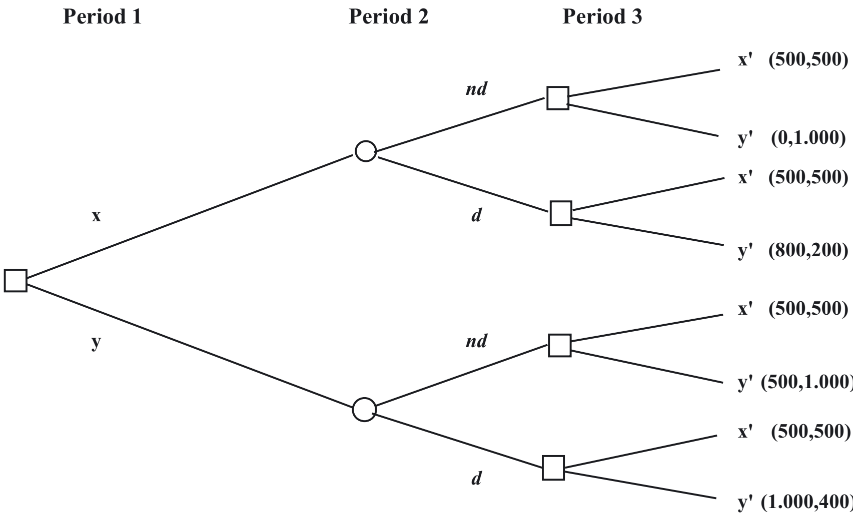

Figure 1: The 3-period decision tree associated to the decision problem

Let us now denote a policy plan by a triple (*a;b,c*), where $a \in \{\mathrm{x}, \mathrm{y}\}$ and *b,c* $\in \{\mathrm{x}', \mathrm{y}'\}$, where *a* stands for the policy adopted in period 1, *b* for the policy that is adopted in period 3 if *nd* prevails and *c* stands for the policy adopted in period 3 if the disaster *d* takes place. Note that these policy plans are contingent plans that prescribe a particular action for any state of the world. This makes it easy to calculate the expected utilities of the two individuals for all the respective policy options, as shown in Table 1.

Let us now consider two simple social welfare functions that are of interest for us, the utilitarian one $W_u : \mathbb{R}^2 \to \mathbb{R}$ defined as: $W_u(U_A, U_B) = U_A + U_B$ and the egalitarian one $W_e : \mathbb{R}^2 \to \mathbb{R}$ defined as: $W_e(U_A, U_B) = \min\{U_A, U_B\}$.

If we now apply these two functions to the expected utilities given in column 2 of Table 1, that is to say if we calculate $W_u(EU_A, EU_B)$ and $W_e(EU_A, EU_B)$ respectively, both take their maximal value with the plan $(\mathrm{y}; \mathrm{y}', \mathrm{y}')$, which will thus be the chosen plan. But suppose now that we embark on this plan and choose indeed option y in period 1 and reconsider the plan in period 3, after the uncertainty has been resolved. This time we apply the welfare functions to the final utilities. As can be verified in Table 2, if we have chosen y in period 1 the following becomes true: whilst W_u chooses y′ in both states of the nature, W_e selects x′ at least in the case of a disaster. This is in contradiction to our initial plan $(\mathrm{y}; \mathrm{y}', \mathrm{y}')$.

society etc. Explicitly modelling all these factors would go beyond the aim of this example. Using this illustrative example just should show, that even in a very simple society the problems shown in the following *possibly can* arise.

Table 1

Expected Utilities and values of the utilitarian and egalitarian welfare function (ex ante evaluation) associated to every policy plan

Policy Plan	(EU_A, EU_B)	$W_u(EU_A, EU_B)$	$W_e(EU_A, EU_B)$
(x; x′, x′)	(500,500)	1.000	500
(x; x′, y′)	(515,485)	1.000	485
(x; y′, x′)	(25,975)	1.000	25
(x; y′, y′)	(40,960)	1.000	40
(y; x′, x′)	(500,500)	1.000	500
(y; x′, y′)	(525,495)	1.020	495
(y; y′, x′)	(500,975)	1.475	500
(y; y′, y′)	(525,970)	**1.495**	525

Table 2

Histories, policies and respective outcomes with their evaluation by the utilitarian and egalitarian welfare function in Period 3

Period 1	Period 2	Policy at Period 3	Outcomes (U_A, U_B)	$W_u(U_A, U_B)$	$W_e(U_A, U_B)$
(x)	(*nd*)	(x′)	(500,500)	1.000	500
(x)	(*nd*)	(y′)	(0,1.000)	1.000	0
(x)	(*d*)	(x′)	(500,500)	1.000	500
(x)	(*d*)	(y′)	(800,200)	1.000	200
(y)	(*nd*)	(x′)	(500,500)	1.000	**500**
(y)	(*nd*)	(y′)	(500,1.000)	**1.500**	500
(y)	(*d*)	(x′)	(500,500)	1.000	**500**
(y)	(*d*)	(y′)	(1.000,400)	**1.400**	400

It seems thus that the utilitarian welfare function is somehow better adapted to dynamic situations than the egalitarian criterion. One might conclude that dynamic consistency and egalitarianism are incompatible. This judgement would be premature though. In fact, there are two questions involved here: The first is whether the two choices before and after uncertainty is resolved are dynamically consistent. We say that an agent chooses in a dynamically consistent way, if she chooses at some time t_n exactly the same option (if still available) as she chose at some time t_{n-h} where the only difference in the set of available options between the two times is that some options available at t_{n-h} are no longer available at t_n. It therefore seems that an agent's decision based on the egalitarian welfare function lacks the property of dynamic consistency.

But we also need to consider a second question, namely whether the welfare evaluation that is employed at the two times is of the type *ex ante* or *ex post.* The ex ante approach is what we applied so far, that is to say, pick a decision node and apply the welfare function to the individual expected utilities at this node. In practical life this comes down to letting the individuals maximise their individual expected utility and then evaluate the outcome according to some social norm. The ex post approach on the other hand consists in first calculating the value of the welfare functions W_u and W_e respectively on basis of the actual utilities in every state, and *then* calculating the expected value of the welfare functions for each plan. If we apply this method in period 1 to the given example we obtain the expected values for the two welfare functions given in Table 3.

Table 3

Expected values of the utilitarian and egalitarian welfare function associated to every policy plan (ex post evaluation)

Policy Plans	$EW_u(U_A, U_B)$	$EW_e(U_A, U_B)$
$(x; x', x')$	1.000	**500**
$(x; x', y')$	1.000	485
$(x; y', x')$	1.000	25
$(x; y', y')$	1.000	10
$(y; x', x')$	1.000	**500**
$(y; x', y')$	1.020	495
$(y; y', x')$	1.475	**500**
$(y; y', y')$	**1.495**	495

Following the utilitarian criterion, we will still choose the plan $(y; y', y')$ and we know that this plan is dynamically consistent. The egalitarian principle on the other hand will either choose $(x; x', x')$, $(y; x', x')$ or $(y; y', x')$, and it is easily checked that W_e would choose x' in period 3 in any case and any state of nature. So the choice in period 3 will be the same as planned in period 1. Applying the ex post approach thus also allows choices based on an egalitarian welfare function to be dynamically consistent. We should call the property that choices based on an utilitarian welfare function possess in addition to dynamic consistency then rather "equivalence of ex ante and ex post evaluation". In the following section in which we turn to a more formal study of these questions we will introduce this property as a Linearity condition on social choice functions.

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3. Some Formal Results on Linearity and Independence

We now need to introduce some notation in order to both, restate a result of Myerson (1981) (in a slightly modified version) and of Mongin (1994) and to state our own corollary result based on the two. It will basically relate Linearity of Myerson choice functions (MCF) to the independence axiom that is well known from von Neumann-Morgenstern (VNM) expected utility theory, but applied here to a multi-profile social choice framework. This is why we need to introduce individual VNM functions, then the rather special kind of MCF's, the formalism of Social Welfare Functionals (SWFL) that aggregate individual VNM utility functions to a social preference relation and we will also make use of an "ordinary" choice function that is based on a binary relation that is an ordering.

Notation

Let $N = \{1, 2, \ldots, n\}$ be a finite set of at least 3 integers and $\Xi = \{x_1, x_2, \ldots x_m\}$ a finite set with at least 2 elements. $p = \{p_1, p_2, \ldots, p_m\} \in \mathbb{R}^m_+$, $\sum_{j=1}^{m} p_j = 1$, is called a lottery and $\mathfrak{L}$ is the set of all lotteries. $v_i : \mathfrak{L} \to \mathbb{R}$ is the VNM-utility function of $i \in N$[3], $\mathfrak{V}$ is the set of all VNM-utility functions and we write $\mathfrak{v}(p) = (v_1(p), v_2(p), \ldots, v_n(p))$ for a vector of VNM utilities. Let further be $\succeq \subseteq \mathfrak{L} \times \mathfrak{L}$ a binary relation on $\mathfrak{L}$ and let $\mathfrak{B}$ be the set of all binary relations on $\mathfrak{L}$. A function $F : \mathfrak{V}^n \to \mathfrak{B}$ is called a Social Welfare Functional (SWFL) and $CP \subseteq \mathbb{R}^n$ is called a choice problem if and only if it is non-empty, closed, convex and comprehensive. $\mathrm{X\Pi}$ is the set of all choice problems, on which we define the so called Myerson choice function (MCF) $f : \mathrm{X\Pi} \to \mathbb{R}^n$. $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is an ordering if it is complete, reflexive and transitive. We call $C : \mathrm{X\Pi} \to \mathbb{R}^n$ the choice function on $\mathrm{X\Pi}$ associated to R and defined as $C(S) \subseteq S$, with $C(S) = \{b \in S | \forall d \in S : bRd\}$, $\forall S \subset \mathrm{X\Pi}$.

The Results

Theorem 1 (Myerson 1981)

Assume that $\mathrm{X\Pi}$ is a convex and compact set of choice problems CP and assume that the MCF $f : \mathrm{X\Pi} \to \mathbb{R}^n$ is linear and satisfies Strong Pareto, then f is utilitarian.

As can easily be verified from the definition of Linearity in Appendix 1, Linearity in this formal framework captures exactly the two notions of ex ante

[3] A binary relation on the lotteries $p, q, r \in \mathfrak{L}$ that satisfies the axioms of ordering, continuity and independence (for definitions see the Appendix) can be represented by such a function $v_i : \mathfrak{L} \to \mathbb{R}$ that takes the so called expected utility form: For $i \in N, x_j \in \Xi$, $p_j \in [0,1]$:

$$v_i(x_1, p_1; \ldots; x_j, p_j) \equiv \sum_{j=1}^{m} U(x_j) p_j \,.$$

and ex post evaluation employed in our numeric example, where the left hand side of the condition refers to the ex ante evaluation and the right hand side represents the ex post approach. Thus, Linearity does nothing else but demanding that the ex ante approach and the ex post approach yield the same results. This is a more demanding property than mere dynamic consistency as our example showed. The result proofs formally what in our example also could have been a result of well chosen numbers: Ex ante – ex post equivalence together with some pareto principle implies utilitarianism. In other words, if we want that the worst off victims get priority in any social risk policy, ex ante preferences of individuals cannot fully be satisfied, since we need to adhere to ex post pareto optimality if we want to keep dynamic consistency. We will discuss this in the conclusions.

Theorem 2 (Mongin 1994)

If a SWFL $F : \mathfrak{V}^n \to \mathfrak{V}$ satisfies Continuity, Independence, IIA and Strong Pareto, then there exists a vector $\mathfrak{a} = (a_1, \ldots a_n) \in \mathbb{R}^n_+$, such that:

$$\forall \mathfrak{v} \in \mathfrak{V}^n, \forall p, q \in \mathfrak{L} : \mathrm{p}F(v)q \Leftrightarrow \sum_{i=1}^{n} a_i v_i(p) \geq \sum_{i=1}^{n} a_i v_i(q) \, .$$

This result is tantamount to saying that in a more sophisticated social choice framework, where individuals have VNM preferences and share their probability evaluations, accepting the independence axiom also for the collective choice under risk implies utilitarianism as well as Linearity did in the different framework of Myerson.

Corollary:

Let $f : X\Pi \to \mathbb{R}^n$ be a MCF; if it can be written as a choice function $C : X\Pi \to \mathbb{R}^n$ that is induced by the relation *R,* which is derived from a SWFL $F : \mathfrak{V}^n \to \mathfrak{V}$ of Mongin, then *f* is linear.

Proof see Appendix 2.

Since the two results are stated in such different formal frameworks we need this corollary result to be able to state that the Linearity condition and the Independence axiom in the two respective frameworks play the same role in guaranteeing a utilitarian social welfare evaluation.

It is important to note that this observation does not touch upon Theorem 2 in Myerson 1981, where he replaces the Linearity condition by concavity (for a definition see Appendix 1). In his theorem 2, Myerson basically states that concavity, weak Pareto and Independence of Irrelevant Alternatives (IIA) guarantees that a MCF is either utilitarian or egalitarian. Myerson is rather cautious in interpreting this second result, but he states that much: *"So, when a concave choice function is used, the timing of social choices can make a difference; but timing would never be a cause for dispute, because all indivi-*

duals would agree that earlier (planned-ahead) choices yield better outcomes." He also gives an example indicating that two individuals would rather select a fair toss between the allocation (0,10) and (10,0) and thus expecting (5,5), than selecting (4,4). Surely this gives ex ante preferences their right. But I doubt that any true egalitarian would agree that this yields an equitable distribution. Clearly it is contradictory to the notion of giving priority to the worst off victims after a disaster.

4. Conclusions

What can we conclude from these results for the question of catastrophic risk mitigation programmes that want to meet the moral intuition that the worst off victims should have priority in getting compensation? The first conclusion that may not be surprising is, that a society with such concerns cannot adhere to the independence axiom. We did not fully examine all possible ways how such a society can still adhere to dynamic consistency. But one such way became obvious and that is the second conclusion: A society that gives priority to worst off victims cannot respect ex ante preferences of individuals with regard to risk. Voluntary individual insurance for losses after catastrophic events will not necessarily guarantee an ex post outcome that provides enough care for the worst off victims. Thus an egalitarian society aiming at caring most for the worst off victims cannot provide the right to individuals to take whatever risk he or she prefers. Concerns for the worst off victims after a catastrophic event lead to limitations for ex ante risk preferences. A society that is known to provide care for its members in case of catastrophic events usually has problems in motivating its individuals to take voluntary insurance for catastrophic events. But even if it could motivate its members to do so, it will still have to allocate some extra funds for the worst off victims.

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Appendix 1

Axioms on the MCF *f* :

Pareto Indifference: A multi-valued function $f : X\Pi \to \mathbb{R}^n$ satisfies Pareto Indifference $\Leftrightarrow$

(i) $f(S) \subseteq S$, and

(ii) $\mathfrak{v} = \mathfrak{w}$ und $\mathfrak{v} \in f(S) \Rightarrow \mathfrak{w} \in f(S), \forall S \in X\Pi, \forall \mathfrak{v}, \mathfrak{w} \in \mathbb{R}^n$.

Strict Pareto: A function $f : X\Pi \to \mathbb{R}^n$ satisfies Strong Pareto $\Leftrightarrow$

(i) $f(S) \in S$, and

(ii) $\mathrm{v_i} \geq f_i(S) \forall i$ and $\exists\, i : \mathrm{v_i} > f_i(S) \Rightarrow \mathfrak{v} \notin S, \forall S \in X\Pi, \forall \mathfrak{v} \in \mathbb{R}^n$.

Strong Pareto: Pareto Indifference & Strict Pareto.

Linearity: A function $f : X\Pi \to \mathbb{R}^n$ is linear $\Leftrightarrow f(\lambda S + (1-\lambda)T) = \lambda f(S) + (1-\lambda) f(T), \forall S, T \in X\Pi, \forall \lambda \in [0,1]$, and $\lambda S + (1-\lambda)T \in X\Pi$.

Where $\lambda S + (1-\lambda)T$ is defined to be the set $\lambda S + (1-\lambda)T = \{\lambda c + (1-\lambda)t | c \in S \text{ and } t \in \mathrm{T}\}, \forall S, T \subseteq \mathbb{R}^n$.

Concavity (not used in either proof or Lemma): A function $f : X\Pi \to \mathbb{R}^n$ is concave $\Leftrightarrow f(\lambda S + (1-\lambda)T) \geq \lambda f(S) + (1-\lambda) f(T), \forall S, T \in X\Pi, \forall \lambda \in [0,1]$, and $\lambda S + (1-\lambda)T \in X\Pi$.

Utilitarianism: A function $f : X\Pi \to \mathbb{R}^n$ is utilitarian $\Leftrightarrow \exists$ a vector $\mathfrak{a}\ (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that:

(i) $\sum_{i=1}^{n} a_i = 1$ and $\forall a_i > 0$,

(ii) $\mathfrak{a} f(S) = \max_{\mathfrak{v} \in S} \mathfrak{a}\mathfrak{v}, \forall S \in X\Pi$.

Axioms on the SWFL *F*:

Continuity: $\forall \mathfrak{v} \in \mathfrak{V}^n, F(\mathfrak{v})$ is continuous $\Leftrightarrow \forall p, q, r \in \mathfrak{L}$, $\{\lambda \in [0,1]$: $pF(\mathfrak{v})\ [\lambda p + (1-\lambda)q]\}$ and $\{\lambda \in [0,1]$: $[\lambda p + (1-\lambda)q] F(\mathfrak{v}) r\}$ are closed subsets of [0,1].

Independence: $\forall \mathfrak{v} \in \mathfrak{V}^n, F(\mathfrak{v})$ satisfies independence that is to say: $\forall p, q, r \in \mathfrak{L}$, $\forall \lambda \in [0,1], pF(\mathfrak{v})\, q \Leftrightarrow [\lambda p + (1-\lambda)q] r F(\mathfrak{v}) [\lambda q + (1-\lambda) r]$.

Pareto-Indifference: $\forall \mathfrak{v} \in \mathfrak{V}^n, \forall p, q \in \mathfrak{L}, \mathfrak{v}(p) = \mathfrak{v}(q) \Rightarrow pI(\mathfrak{v})q$, where $\mathrm{I}(v)$ is the symmetric part of the relation $F(v)$.

Strict Pareto: $\forall \mathfrak{v} = (v_1, \ldots, v_i, \ldots, v_n) \in \mathfrak{V}^n, \forall p, q \in \mathfrak{L},\ v_i(p) \geq v_i(q), i = 1, \ldots, n$ & $\exists j : v_j(p) > v_j(q) \Rightarrow pP(\mathfrak{v})q$ where $P(v)$ is the asymmetric part of the relation $F(v)$.

Strong Pareto: Pareto-Indifference & Strict Pareto.

Independence of irrelevant alternatives (IIA): $\forall \mathfrak{v},\ \mathfrak{v}^* \in \mathfrak{V}^n, \forall p, q \in \mathfrak{L},\ \mathfrak{v}(p) = \mathfrak{v}^*(p)$ & $\mathfrak{v}(q) = \mathfrak{v}^*(q) \Rightarrow pF(\mathfrak{v})q$ if and only if $pF(\mathfrak{v}^*)q$.

Appendix 2

Lemma 1: There exists a relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ defined by: $\forall (b, d) \in \mathbb{R}^n \times \mathbb{R}^n, \mathrm{b}Rd \Leftrightarrow \exists \mathfrak{v} \in \mathfrak{V}^n, p, q \in \mathfrak{L} : \mathfrak{v}(p) = b, \mathfrak{v}(q) = d, xF(\mathfrak{v})y$ that is an ordering and satisfies the axioms of Continuity, Independence, and Strong Pareto.

Proof of Lemma 1: see Mongin (1994, Lemma 2, p. 339).

Lemma 2: Let $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be an ordering and let $C : \mathrm{X}\Pi \rightarrow \mathbb{R}^n, C(B) \subseteq B$, with $C(B) = \{b \in B | \forall d \in B : bRd\}, \forall B \in \mathrm{X}\Pi$ be its associated choice function. If R satisfies independence, then C is linear in a generalised meaning.

Proof of lemma 2: We need to show that $C(\lambda B + (1-\lambda)D) = \lambda C(B) + (1-\lambda)C(D)$.

1) $C(\lambda B + (1-\lambda)D) \subseteq \lambda C(B) + (1-\lambda)C(D)$:

$C(\lambda B + (1-\lambda)D) = \{e \in \lambda B + (1-\lambda)D | \forall e' \in \lambda B + (1-\lambda)D, eRe'\}$

$C(\lambda B + (1-\lambda)D) =$

$= \{e$ of the form $\lambda b + (1-\lambda)d$ for $b \in B, d \in D | \forall e'$ of the same form, $eRe'\}$.

By independence we have:

$\lambda b + (1-\lambda)dR\lambda b' + (1-\lambda)d$ for $b, b' \in B, d \in D \Leftrightarrow bR\,b'\,\forall b' \in B \Rightarrow b \in C(B)$ and: $\lambda b + (1-\lambda)d\,R\,\lambda b + (1-\lambda)d'$ for $b \in B, d \in D \Leftrightarrow dR\,d'\,\forall d' \in B \Rightarrow d \in C(D)$. Thus, e can be written as an element of $\lambda C(B) + (1-\lambda)C(D)$. q.e.d.

2) $C(\lambda B + (1-\lambda)D) \supseteq \lambda C(B) + (1-\lambda)C(D)$:

Let $g \in \lambda C(B) + (1-\lambda)C(D)$, thus $g = \lambda b + (1-\lambda)d$ for $b \in B, d \in D$, thus $b \in C(B)$ and $d \in C(D)$ and from this we have $bRb'\,\forall b' \in B$ and $dRd'\,\forall d' \in D$.

From independence we have: $bRb' \Leftrightarrow \lambda b + (1-\lambda)dR\lambda b' + (1-\lambda)d$ and: $dRd' \Leftrightarrow \lambda b' + (1-\lambda)d\,R\,\lambda b' + (1-\lambda)d'$.

Thanks to Transitivity of R we have: $\lambda b + (1-\lambda)d\,R\lambda b' + (1-\lambda)d$ and $\lambda b' + (1-\lambda)d\,R\,\lambda b' + (1-\lambda)d' \Rightarrow \lambda b + (1-\lambda)d\,R\,\lambda b' + (1-\lambda)d'\,\forall b' \in B, \forall d' \in D$.

Thus, $g \in C(\lambda B + (1-\lambda)D)$. q.e.d.

Lemma 2 is jointly established by statements 1) and 2). The reasoning was made for $\lambda \neq 0$, the case $\lambda = 0$ is trivial.

Lemma 3: Let there be given a preference relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ that is an ordering and satisfies the axioms of Continuity, Independence and Strong Pareto. Then the choice function associated to it, $C : \mathrm{X}\Pi \rightarrow \mathbb{R}^n$, is utilitarian in a generalised meaning, that is to say, there exists a vector $a = (a_1, \ldots, a_n) \in R^n$ such that:

(i) $\sum_{i=1}^{n} a_i = 1$ and $\forall a_i \geq 0$, and

(ii) $w \in C(B) \Leftrightarrow aw \geq aw', \forall w' \in B$.

Proof of lemma 3: Applying the expected utility theorem (as defined in Fishburn 1982, chap. 1) we can conclude that R can be represented by a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V = \sum a_i v_i + b$. That comes down to saying that there exists a vector $\mathfrak{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_+$ such that $\forall \mathfrak{v}, \mathfrak{v}' \in \mathbb{R}^n$, $\mathfrak{v} R \mathfrak{v}' \Leftrightarrow \sum_{i=1}^n a_i\, v_i \geq \sum_{i=1}^n a_i\, v_i'$. Thus, the choice function induced by R satisfies (ii). q.e.d.